Functions and Calculus

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1. INTRODUCTION

One purpose of the function is to represent *how things change*. With this meaning it is natural to move on to consider the calculus concepts of the *rate of change* (differentiation) and *cumulative growth* (integration) together with the remarkable fundamental theorem of calculus that tells us that differentiation and integration are essentially inverse processes.

The calculus traditionally focuses on mastery of symbolic methods for differentiation and integration and applying these to solve a range of problems. It is both a climax of school mathematics and a gateway to further theoretical developments. This position between elementary and advanced mathematics allows it to be approached in different ways, with a consequent variety of curricula. In some countries calculus is studied in an intuitive form in school, with the limit concept introduced dynamically in terms of a variable quantity 'getting close to' a fixed limiting value. In others the focus is turned towards the formal theory of mathematical analysis starting from a formal ε–δ (or equivalent) definition of the limit. Other curricula occupy a midway position, building on both intuitive ideas but also highlighting formal definitions.

Mamona-Downs (1990) compared the two extremes—the intuitive form in British schools and the logical form in Greek schools and confirmed previous research that the intuitive approach gives side-effects which clash with the formal definition in ways which will be discussed later in this chapter, whilst showing that the formal approach emphasises the logic but gives less conceptual insight.

Traditional American calculus texts occupy the middle ground, with students usually meeting the calculus for the first time at college where it is considered appropriate to include aspects of the formal theory. Here the vast financial rewards available when a text is widely adopted has the effect that no book leaves out anything important contained in a competitor, so calculus texts all grow to enormous size. Most traditional calculus books include all the topics an instructor might wish to teach, with a large number of worked examples and exercises to satisfy the most anxious student. Even so, the heavy diet of procedural exercises produced failure rates between 30% and 50% (Anderson & Loftsgaarden 1987; Peterson 1987). Despite isolated attempts at trying something really new, such as Keisler's (1976) pioneering work to introduce an intuitive infinitesimal

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approach, the American system began to reach a position of gridlock. In the fateful year 1984, E. E. Moise wrote:

> For the overwhelming majority of students, the calculus is not a body of knowledge, but a repertoire of imitative behaviour patterns.

In that same Orwellian year, the *American Mathematical Monthly* carried a full page advertisement for the computer algebra system *MACSYMA* which

> … can simplify, factor or expand expressions, solve equations analytically or numerically, differentiate, compute definite and indefinite integrals, expand functions in Taylor or Laurent series.

Suddenly the whole rationale of the calculus became questioned—if computer software can do all the things that a student is required to do on a calculus examination, why do they need to learn to do it anyway?

Although mainframe developments of such symbol-manipulators occurred almost exclusively in the USA and Canada, the late 1970s and early 1980s saw personal computers being launched in educational projects all over the world. School curriculum builders were beginning to investigate the use of computers in mathematics in general and calculus in particular. At first often all that was available was a computer and a programming language such as BASIC in which developers began to write numerical algorithms, encouraging students to do the same. Then in the early 1980s high resolution graphics arrived and graphical software for calculus began being written in profusion.

In 1985 the first real practical symbol manipulator arrived on a personal computer in the form of *MuMath*, to be superseded by its later, more user-friendly re-incarnation, *Derive* (Stoutemyer et al. 1985; 1988).

With the proliferation of new computer approaches to the calculus and the perceived log-jam of traditional calculus, the 'Calculus Reform Movement' began in the USA. The quest for reform to *A Lean and Lively Calculus* (Douglas 1986) was answered by a rallying call in *Calculus for the New Century: A Pump, Not a Filter* (Steen 1988) to make calculus a genuine driving force for learning instead of a filter which weeded out poor students. Subsequent developments were all stimulated in varying degrees by the use of new technology.

The technological revolution brought with it new market-driven factors, with large companies cooperating with educators to develop new tools. The costs of delivering the calculus curriculum were escalating and new assessment methods were being considered to take account that students now solved many calculus problems using technological supports.

The most economically viable approaches in recent times use graphic calculators with graphic, numeric and subsequently symbolic facilities. Though they lack the full facilities of a desk-top computer, they have the great advantage of portability, allowing the student to use them anywhere and at any time. Meanwhile, computer algebra systems, previously only available on mainframes, migrated to personal computers, leading to sofware such *Mathematica*, *Maple*, *Reduce*, *Theorist* and *MathCad*. In varying degrees these offer environments for presenting curriculum material, writing reports, performing calculations, manipulating symbols, drawing a range of graphs, and programming new extensions at the desire of the user. Many practising mathematicians found them to be a creative paradise for new experimental research and in turn they offered a wide range of possibilities in teaching and learning calculus.

At the same time the motivating factors behind the reforms were many and varied. They included altruistic desires to make calculus more understandable for a wider range of students, commercial desires to produce saleable products, practical considerations of what actually needed to be taught, reflection on the type of mathematics that is suitable in a technological age, and a growing aspiration to research the learning process to understand how individuals conceptualise calculus concepts.

2. COGNITIVE CONSIDERATIONS

2.1 The Role of Learning Theories in the Calculus Reform

A characteristic of recent developments has been a focus of attention not only on the mathematics to be taught, but also the mental processes by which it is conceived and learnt (e.g. Dubinsky 1992). Kaput (1992; 1993) and Nemirovsky (1993) focus on the way that younger children have intuitive sense of concepts such as distance, velocity, acceleration, which can be utilised in conjunction with computer simulations to study aspects of calculus at a far earlier age. Moreover, the simulations involved, such as driving a car along a highway—linked to numeric and graphic displays of distance and velocity against time—allow a study of change which is not limited to functions given by standard formulae (figure 1).

This widens the representations available in the calculus to include:

- *enactive* representations with human actions giving a sense of change, speed and acceleration,
- *numeric* and *symbolic* representations that can be manipulated by hand or by computer, including the possibility of programming by the student,
- *visual* representations that can be produced roughly by hand or more accurately and dynamically on computers,

and

- *formal* representations in analysis that depend on formal definitions and proof.

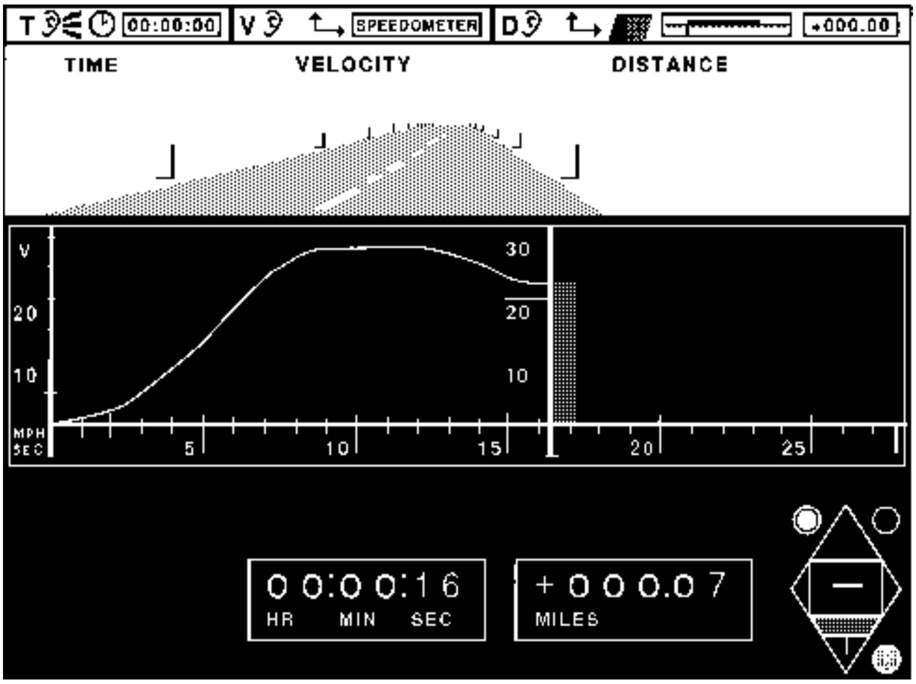


Figure 1: MathCars: simulating relationships between time, distance and velocity

A diagrammatic representation of the growth of representations and the building of the concepts of calculus may be formulated as in figure 2.

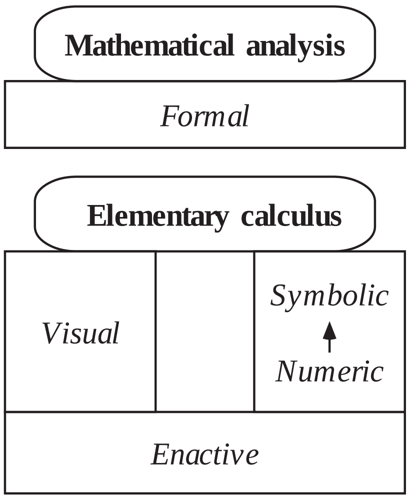


Figure 2: Representations in calculus/analysis

This diagram carries an explicit theoretical viewpoint—that enactive experiences provide an intuitive basis for elementary calculus built with numeric, symbolic and visual representations, but that mathematical analysis requires a higher level of formal representation.

The cognitive growth by which this occurs requires significant constructions and re-constructions of knowledge. For instance, enactive sensations of moving objects may give a sense that ‘continuous change’ implies the existence of a ‘rate of change’, suggesting an equivalence between the theoretically different formal concepts of continuity and differentiability. More significantly, the formal definitions and theorems of analysis require subtly different cognitive qualities which are almost certainly inappropriate in a first course in calculus. This suggests a fundamental fault-line in ‘calculus’ courses which attempt to build on formal definitions and theorems from the beginning.

The way in which numeric and symbolic representations develop involves an interesting form of cognitive growth. There are recurring cycles of activity in which a *process*, such as counting, becomes a *concept*, such as number. Other instances include the process of addition becoming the concept of sum, the process of equal sharing becoming the concept of fraction, the process of calculating ratio becoming the concept of rate, and the limiting process becoming the concept of limit.

Various authors, including Piaget (1972), Dubinsky (1991) and Sfard (1991), theorise that the growth of human knowledge starts with actions (first on the environment), some of which become repeatable processes and are later conceived as objects in their own right to be manipulated on a higher level by further mental processes. Gray and Tall (1994) name the combination of *pro*cess and con*cept* produced by the process which may both be evoked by the same symbol, a *procept*. This proves to be particularly apposite in the study of the calculus because *function*, *derivative*, *integral* and the fundamental *limit* notion are all examples of procepts. Indeed, the theory of functions and calculus can be summarised in outline as the study of the ‘doing’ and ‘undoing’ of the processes involved (figure 3).

Procept		
Change: FUNCTION	doing	calculating values
	undoing	solving equations
Rate of change: DERIVATIVE	doing	differentiation
	undoing	anti differentiation, solving differential equations
Cumulative growth: INTEGRAL	doing	integration
	undoing	fundamental theorem of calculus

Figure 3: Three procepts in functions and calculus

The various representations each have their own characteristics that offer potential cognitive advantages and disadvantages for representing the underlying limit procept and for ‘doing’ and ‘undoing’ each of the procepts of function, derivative and integral. These include using visual ideas for conceptual insight, numerical computations for practical experience, symbolic manipulations performed by the computer to support those with limited facility in algebraic manipulation, and programming of various kinds to encourage the student to construct procedures on the computer to represent calculus concepts.

Modern calculus reforms (both within the USA and elsewhere) seek to use these representations to make the subject more practical and meaningful. Cottrill, Dubinsky et al. (1995) use the principle that *actions* become repeatable *processes* which are encapsulated as *objects* and then related in a wider *schema*, in a theoretical development given the acronym APOS. This is realised by programming activities which place visualisation at the end of the line as a visual representation of the function constructed through encapsulating a programming process as an object. Tall and Sheath (1983) see the visualisation of the gradient of the graph as an early stage in the calculus to be linked to numeric and symbolic representations. Here the visualisations are focused mainly on *graphical* representations of the function as a graph. Gleason et al. (1990) produce a course based as much on mathematical belief as on cognitive growth, in which:

> One of the guiding principles is the 'Rule of Three,' which says that wherever possible topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and where students see each major idea from several angles. (Hughes Hallett 1991, p.121)

There is therefore a spectrum of possible approaches to the calculus, from real-world calculus in which intuitions can be built enactively using visuo-spatial representations, through the numeric, symbolic and graphic representations in elementary calculus and on to the formal definition-theorem-proof-illustration approach of analysis which is as much concerned with *existence* of solutions as with their actual construction. (Figure 4.)

Faced with such an array of possible approaches, certain broad differences should first be noted. Traditional calculus used to be a mixture of manipulative symbolism and qualitative visualisation with possible deductive elements from analysis. The computer allows not only a numeric quantitative approach to carry out the many calculations required in, say, computing the approximate area under a curve, it also allows graphical representations to be drawn at the will of the user, offering a possible conceptual approach based on visualisation. In turn this broadens the approach to the calculus by allowing functions to be defined by data or numeric procedures, with solutions of differential equations given in numeric and visual form where there may be no solutions in terms of standard formulae.

Underlying all these approaches to the calculus, however, is the limit concept which research shows to have deeply embedded cognitive difficulties (summarised in Cornu 1992). The enactive real-world approach

		Representations				
		Visuo-spatial	**Numeric**	**Symbolic**	**Graphic**	**Formal**
Procepts		*Enactive* observing experiencing	*Quantitative* estimating approximating	*Manipulative* manipulating limiting	*Qualitative* visualizing conceptualizing	*Deductive* defining deducing
Change: FUNCTION	doing	distance, velocity etc. changing with time	numerical values	algebraic symbols	graphs	set-theoretic definition
	undoing	solving problems	numerical solutions of equations	solving equations symbolically	visual solutions where graphs cross	intermediate value & inverse function theorems
Rate of change: DERIVATIVE	doing	velocity from time-distance graph	numerical gradient	symbolic derivative	visual steepness	formal derivative
	undoing	solving problems e.g. finding distance from velocity	numerical solutions of differential equations	antiderivative —symbolic solutions of differential equations	visualize graph of given gradient	antiderivative—existence of solutions of differential equations
Cumulative growth: INTEGRAL	doing	distance from time-velocity graph	numerical area	symbolic integral as limit of sum	area under graph	formal Riemann integral
	undoing	computing velocity from distance	know area—find numerical function	Symbolic Fundamental Theorem	know area – find graph	Formal Fundamental Theorem
		REAL-WORLD CALCULUS	THEORETICAL CALCULUS			ANALYSIS

Figure 4: A spectrum of representations in functions and the calculus

deals with this at a practical approximation level. The graphical approach allows the limit notion to be handled *implicitly*, for instance, by magnifying the graph to see it looking 'locally straight' so that the gradient required is that of the visibly straight graph. This helps the move into elementary calculus without stumbling over the limit concept, but the deficiency may need to be addressed at a later stage when further cognitive reconstruction may prove necessary to cope with formal concepts.

At the formal end of the spectrum there is a wide conceptual gulf between practical calculations or symbol manipulations in calculus and the theoretical proof of existence theorems in analysis. I conjecture that this is so wide that it causes a severe schism in courses (particularly in 'college calculus') which attempt to bridge the gap between calculus and analysis during the first encounter with the subject. This is implicitly recognised by reforms which include only those topics which are found to be essential. For instance, *ProjectCalc* at Duke University (Smith & Moore 1991) found that this meant that 'numerical algebra was in, but the mean value theorem was out' (as quoted in Artigue & Ervynck 1993, p.92). It goes without saying that the mean value theorem inhabits the realms of existence theorems in analysis and sits uncomfortably in the computations of elementary calculus. This implicitly underlines the difficult chasm between elementary calculus and formal analysis.

2.2 Students

Students taking calculus courses cover a wide range of background knowledge, ability and motivation. This complicates not only the design of the curriculum, but also the interpretation of research evaluating its effectiveness. The problem in the calculus is highlighted by the fact that some students appear to make connections and others do not. Thus, a course which is designed to give greater insight by making connections may be a positive help for some and a failure for others.

Krutetskii (1976, p.178) performed a wide range of studies on 192 children selected by their teachers as 'very capable' (or 'mathematically gifted'), 'capable', 'average' and 'incapable'. He found that the gifted children remembered general strategies rather than detail, curtailed their solutions to focus on essentials and were able to provide alternative solutions. Average children remembered more specific detail, shortened their solutions only after practice involving several of the same type, and generally offered only a single solution to a problem. Incapable children remembered only incidental, often irrelevant detail, had lengthy solutions, often with errors, repetitions and redundancies, and were unable to begin to think of alternatives.

He also found that 'giftedness' was manifested in different ways. Of his 34 'gifted' children, 6 were classified as 'analytic', 5 as 'geometric' and 23 as 'harmonic', exhibiting a spectrum of relative preferences for verbal-logical and visual thinking.

Students taking calculus are usually 'capable' or 'gifted', but with wider access, they include many of 'average' ability and below. Given the wide possible spectrum of approaches by such a range of students, it becomes evident that methods that may be essential to some may be inappropriate for others. For example, the repetition of regular problems which seems necessary for curtailment of solution processes for the average student may be less necessary for the gifted, whilst causing inflexible procedural orientations in others. Meanwhile, the flexibility in switching from one representation to another, which seems a characteristic of gifted 'harmonic' thinkers, may prove difficult for the average student. A growing number of research studies report students having difficulties relating representations, and others show some students moving from one representation to another but failing to move flexibly back and forth.

In considering research into the calculus it is well to have such a spectrum of student possibilities in mind, for research on one group of students in one context may well turn up different characteristics from another situation without the two necessarily contradicting each other. There is also the chicken-and-egg problem: are students gifted because they 'have' certain abilities, or do they *become* gifted because they 'develop' these abilities? Such questions dig deep into theoretical and philosophical issues which have their roots in the development of students long before they begin to study the calculus. We now turn our attention to curriculum issues, cognisant of the differences that may occur in the students for whom the curriculum is being designed.

3. FUNCTIONS

3.1 The Function Concept

The notion of function came to prominence first in the writings of Leibniz in the late seventeenth century where he used the term *functio* to describe a variable y whose value depended on a changing variable x. Initially it was conceived as having an explicit formula such as $y=x^2$ and, in the next century, this was denoted by the more general formulation $y=f(x)$. From the beginning a function in the calculus was linked to its graph G—the set of points $(x, f(x))$ in the cartesian plane.

In the twentieth century, the visual idea of the graph $f:A\rightarrow B$ became considered as a set of ordered pairs $\{(x, y) \in A\times B \mid x \in A, y=f(x)\}$, giving the possibility of a set-theoretic definition. A function may now be *any* set G of ordered pairs $G = \{(x, y) \in A\times B \mid x \in A, y \in B\}$, provided only that for every $x \in A$ there is a $y \in B$ such that $(x, y) \in G$ and that this y is unique (if $(x, y_1), (x, y_2) \in G$ then $y_1 = y_2$).

Such a development is not without its conceptual difficulties and cognitive struggles. Sierpiń ska (1992) described how the subtle changes in meaning were accompanied by difficult conceptual obstacles that needed to be overcome. For instance, an individual whose experience of functions in terms of formulae and computation will find it difficult to accept a definition which does not involve these attributes. Sfard (1992) indicated how the *operational* view of mathematics in terms of processes to be carried out seems inevitably to precede the *structural* view using objects and formal definitions, both in history and cognitive development.

Although the set theoretic definition proved highly successful in the systematic formulation of mathematics, it was less successful when adopted for teaching purposes in the 'New Mathematics' curricula of the 1960s. Although students were *told* that a function had a domain D, a range R and a set of ordered pairs (x, y) with $x\in D$, $y\in R$, what they *experienced* was a *formula* such as $y=x^2$ or $f(x)=\sin x+\cos x$. Surveying the function concept in Malaysia, Bakar (1991) found the set-theoretic concept was *in principle* at the root of the curriculum in every year from the age of 11 onwards. But *in practice* the functions were first linear, then quadratic, then polynomial and later rational, trigonometric, exponential or logarithmic. In service courses where the formal definition has less emphasis, memories of it decline markedly in engineers and others studying mathematics by the last year of university.

It is valuable to distinguish carefully between the formal mathematical concept specified by a concept definition and the wider 'concept image' including 'all the mental pictures and associated properties and processes' related to the concept in the mind of the individual (Vinner & Hershkowitz 1980; Tall & Vinner 1981; Dreyfus & Vinner 1982). Vinner (1983) found that, even students who could give a correct set-theoretic definition of a function were likely to use their intuitive images in answering questions about functions. Around 40% of the high school students he tested thought that the graph of a function should have other properties, such as being regular, persistent, reasonably increasing, etc., whilst many did not think the graph in figure 5 is a function.

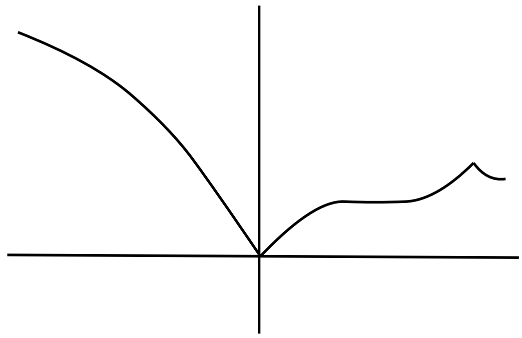


Figure 5: A unfamiliar graph without an obvious formula

He found many students thinking that a function should be given by a single formula, or, if two rules were given their domains should be half lines or intervals.

Markovits et al. (1986; 1988) showed that if students were asked how many functions could be drawn through given points, then two points A, B often evoked only a single straight line through them because 'two points can be connected by only one straight line', whilst the second graph in figure 6 may be considered not to allow a function at all because there seems to be two sets of points on two distinct lines.

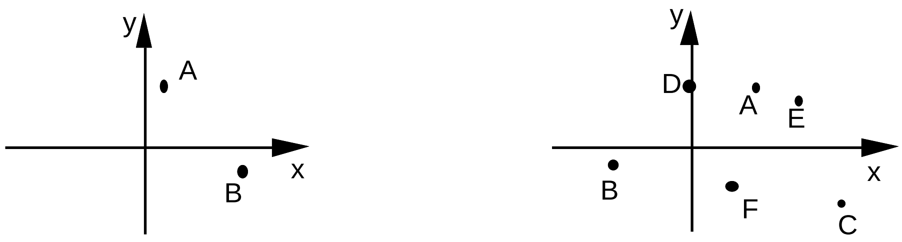


Figure 6: How many graphs can you draw which pass through these points?

Barnes (1988) found that a majority of grade 11 school students and university students did not regard $y=4$ as a function, because it does not *depend* on x, but that $x^2+y^2=1$ *is* a function because it is familiar. Bakar and Tall (1992), Ferrini-Mundy and Graham (1994) and others all found that students had various specific conceptions of a function: that it was given by a formula, that if y was a function of x, it must include x in the formula, that its graph was expected to have a recognisable shape (e.g. polynomial, trigonometric or exponential), and that it was to have certain 'continuous' properties. These had idiosyncratic meanings, for instance, 'continuous' might mean that the graph 'continues', so that a quadrant of a circle $y=\sqrt{(1-x^2)}$ $(0\le x\le 1)$ would not be allowable for some students because this graph should be 'continued' to give a fuller curve (figure 7).

Such concept imagery is of course not confined to students. It clearly occurs in all areas of human endeavour, including the historical development of mathematics where individuals have concept images related to their experience in the prevailing culture. Successive generations do not replace all old elements with new, instead they retain aspects that prove useful and graft on more powerful aspects. So it is that mathematicians and teachers retain dynamic ideas of changing variables alongside formal notions of ordered pairs in a potent but perplexing composite of ideas to pass on to the next generation.

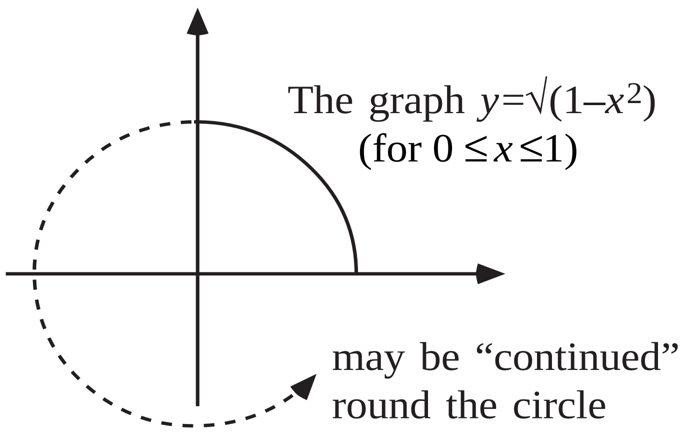


Figure 7: 'Continuing' the graph of a function

3.2 Computer approaches to functions

The computer provides a new environment to explore the function concept. Cuoco (1994) found that an approach to functions through programming in Logo gave significantly different insights from a traditional approach. Students using paper and pencil drawings of graphs saw them as geometric shapes rather than a process of inputting x and outputting y. On the other hand, those programming in Logo not only saw the relationship in input-output terms, they were able to think of a function as an object in its own right. Similar conclusions have been found in structured BASIC which incorporates procedural functions (Li & Tall 1993) and in ISETL (Breidenbach et al. 1992; Cuoco 1994). ISETL (Interactive SET Language) has a further advantage in that the name for a function can be used as an input for another function, hence enhancing its object status.

This shows that the traditional notion of a function represented by a formula and its graph is cognitively different from the notion of function as defined set-theoretically and different again from that conceived in terms of process–object encapsulation.

When software is used to represent function concepts, this is usually done graphically, sometimes with a facility to represent them in tabular form. The way in which the graph is often drawn as a curve may cause students to see it as a whole object. Some programs, such as *RandomGrapher* (Goldenberg et al. 1992) plot random function values to build the graph as a collection of points (figure 8). Although this gives a set of points of the form $(x, f(x))$, further activities may be necessary to see the function process assigning to each value of x the value $y=f(x)$.

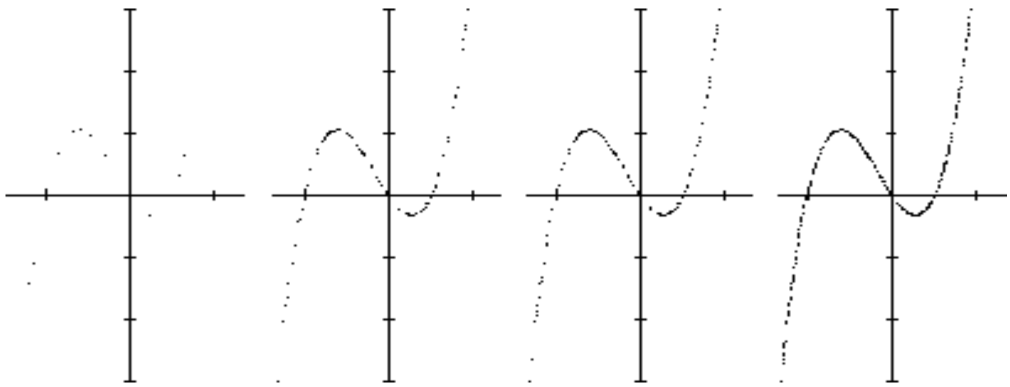


Figure 8: Successive pictures building up a random plot as a collection of points

Other programs link with alternative forms of representation, for instance, *Function Probe* (Confrey 1992) allows graphs to be manipulated *enactively*, using the mouse to transform graphs by translating, stretching, reflecting. Such an approach treats the graph as a single object to be transformed.

Figure 9 shows a problem to transform the top left parabola to give the large inverted parabola. The first move has been made, translating the upper left parabola to the right.

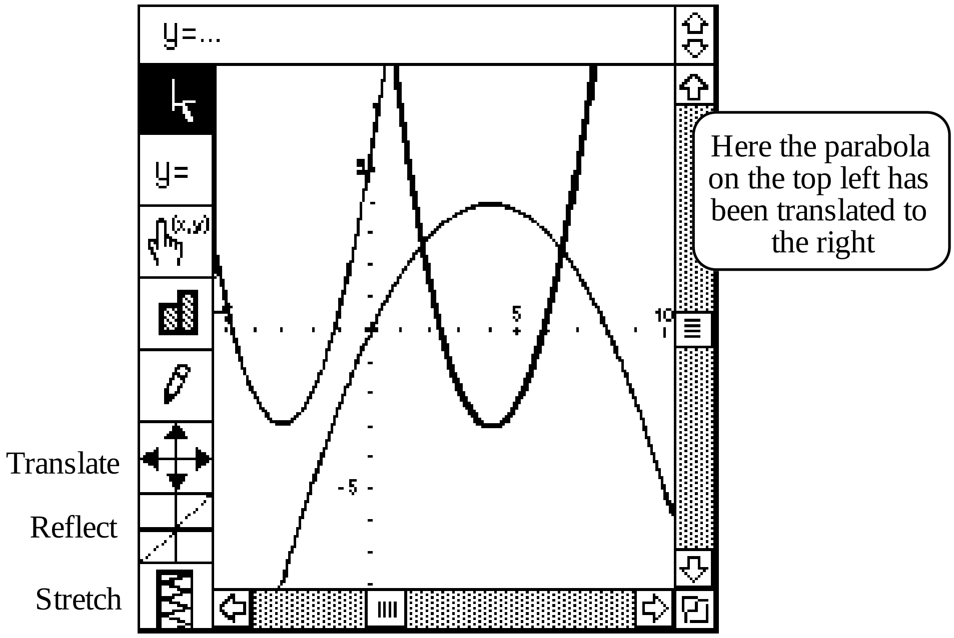


Figure 9: Starting a sequence of moves to translate the top left parabola to the larger inverted parabola

Such software highlights the problems in interpreting the meaning of translations, for instance a horizontal shift to the right by a constant $+c$ changes the graph $y=f(x)$ to $y=f(x-c)$ causes great difficulties for students (Dreyfus & Eisenberg 1987). This is not only a problem linking enactive and symbolic, but also symptomatic of subtle underlying difficulties often hidden in the mathematical theory (Smith & Confrey 1994). (In this case, shifting the graph to the right is equivalent to shifting the domain to the left, and the change in the function symbolism corresponds to the latter rather than the former.)

3.3 Graphic Calculators

Graphic calculators provide a combination of calculator, numeric programming language and graphical output. More recent models provide a growing number of other facilities, such as symbol manipulation, spreadsheet facilities, data handling. These little tools can be carried around in and out of the classroom and their very availability has caused them to be included in a wide variety of courses.

Demana and Waits saw their value as tools and immediately began constructing courses to use them at Ohio State University. Meanwhile, in the UK, the School Mathematics Project incorporated graphic calculators into their calculus course because they were in the hands of the students who were already making use of the facilities.

There have been powerful claims for multiple representations:

> the feature of computers that has recently caused most excitement amongst mathematics educators is the ease of moving from one form of information representation (numerical, graphic and symbolic) to another as the user searches for conceptual understanding and problem solutions.
>
> (Fey 1989, p.255)

But is the power being used to its full potential? Keller and Hirsch (1994) found that students often showed a strong preference for one particular representation (which they quantified in terms of a statistically significant frequency of selection from tables, graphs and equations, using a χ^2 test ($p<0.05$)). (Figure 10.)

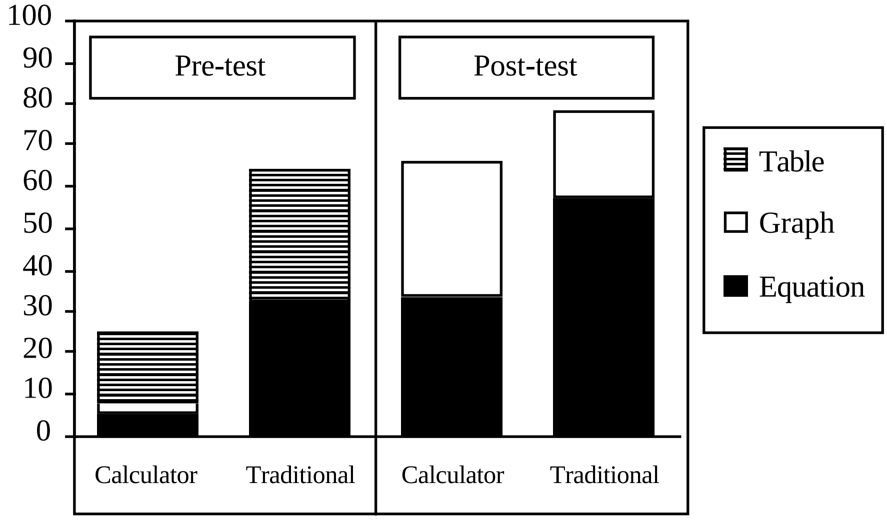


Figure 10: Significant sttudent preferences for specific representations

The students were given a free choice of which course to follow, those choosing the trational route including far more students preferring symbolic representations and both showing a significant number of students preferring to use tables. After the courses, *none* in either group preferred tables, with both groups showing increased preferences for graphs and symbols, and the traditional course becoming even more symbolically biased.

It was also noted that when the problems were purely mathematical with no specific application, more students preferred to use symbols, whilst in problems involving a specific application context, preferences for symbols decreased and graphs were more likely to be selected.

Hart (1991) also reported that students using supercalculators showed definite preferences for certain representations:

- Students confident in symbol manipulation skills tend to use alternate representations only when unsuccessful at finding an answer symbolically,
- students who do not have access to a graphing calculator do not typically choose to use the graphical representation even when it is provided,

- traditional students were more likely to rely on a symbolic representation to solve problems without considering any other possibilities.

(Hart 1991, quoted in Beckmann 1993, p.110)

Furthermore, students who were not confident in symbol manipulation were more likely to use their calculator. When a solution was found, it was rarely checked by using other representations, even when it was wrong. Nevertheless, experimental students showed greater conceptual understanding than traditional students and there was ample evidence that success on the course was not correlated with previous grades so that

> students who might be termed as 'symbolically illiterate'; can be successful in learning and understanding calculus through the use of graphic and numeric tools. (Beckmann 1993, p.112).

The tendency for many students to prefer certain representations can produce unforeseen results. For instance, Caldwell (1995) expected students to find the roots and asymptotes of the rational function

$$f(x) = \frac{x(x-4)}{(x+2)(x-2)}$$

by algebraic means, only to be given a substantial number of approximate solutions such as 0.01 and 3.98 using a graphing calculator. Here a link to a graphical representation was made, without relating back to the precision of the algebra.

Boers and Jones (1993) report students use of a graphic calculator to draw a graph of

$$f(x) = \frac{x^2 + 2x - 3}{2x^2 + 3x - 5}$$

which has a removable discontinuity at x=1. They found that more than 80% of the students had difficulty reconciling the graph with the algebraic information, for example, drawing an asymptote suggested by the zero in the denominator, despite the graphic evidence of the calculator (figure 11).

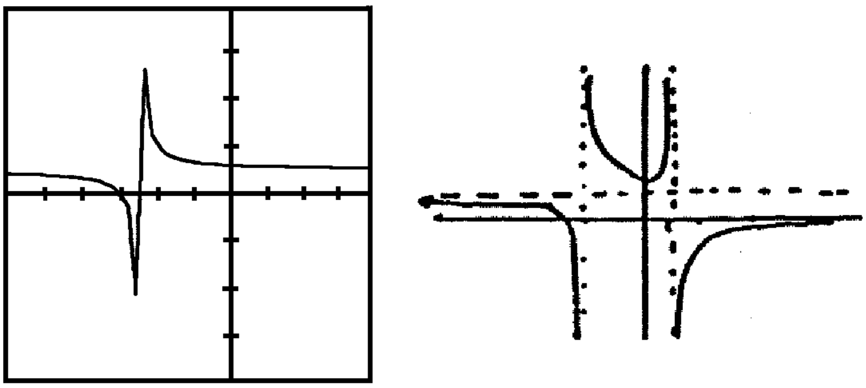


Figure 11: Graphic calculator display and student graph

Some experiments show significant changes when using graphic calculators over a succession of courses. Quesada (1994) introduced graphics calculators into a pre-calculus course where previously an average of 60% of the students finished with a grade D or F, or withdrew from the course. They ‘did not have a clear understanding of the basic families of functions’, ‘could not read basic graphs’ and ‘had not developed basic study habits’. To encourage the use of algebraic skills, there was a policy that no marks were given for a graphic or decimal solution if an algebraic solution was possible. Over three courses, the number of experimental students obtaining D, F, or withdrawing was 43% compared with 69% in the control group. Of the totals taking the final examination, 53% of the experimental students obtained A or B compared with 19% of the control students. When the students moved on to calculus courses, the experimental students again obtained substantially larger percentages of grades A and B in Calculus I and II, though the position was marginally reversed in Calculus III (Quesada 1995).

In these various research studies we see the use of technology giving alternative ways of approaching the function concept with accompanying advantages and disadvantages. Used imaginatively under student control there is evidence of greater student involvement and less likelihood of withdrawal, but there is also evidence of idiosyncratic interpretation of the computer’s representations. Whilst gifted students may have the ability to interchange between representations and focus on the most relevant information, capable students may also benefit significantly from the power of the software, yet use the available facilities in less flexible ways.

4. Limits and Real Numbers

4.1 Cognitive difficulties with the limit concept

At the gateway to the calculus stands the limit concept which must be handled either explicitly or implicitly. Explicitly it is usually treated in terms of considering expressions such as

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}.$$

In ‘intuitive’ terms this may be considered by varying h dynamically to see what happens as $h\to 0$. For $h\neq 0$ it simplifies to $2x+h$, and as h ‘tends to zero’, this expression visibly becomes $2x$. However, this has various hidden problems. The *language* used, with terms like ‘*tends to*’ or ‘*approaches*’ or ‘*has a limit*’ all suggest that the expression *gets close to* the limit, *but can not equal it*. (Schwarzenberger & Tall 1978). The fact that the simplification can only be done for $h\neq 0$, yet to obtain the limit one puts $h=0$ also contributes to the conflict. Students invariably see the limit as a *process* and find it difficult to encapsulate as a limit *concept*. Instead of conceiving of the limit as a specific value to be found, students may focus on the idea of a tiny quantity h that is ‘as small as one desires’, encapsulating this idea as a kind of ‘cognitive infinitesimal’ (Cornu 1992).

Williams (1991) found that a student might use different conceptions of limit selected according to the particular context being considered, without being concerned about possible overall consistencies:

> And I thought about all the definitions that we deal with, and I think they’re all right—they’re all correct in a way and they’re all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions, but it’s like talking about infinity or God, you know. Our mind is only so limited that you don’t know the real answer, but part of it. (Williams 1991, p.232)

When ten students were selected with concept images of a limit at variance with the definition (such as ‘gets close to, but does not reach’), a series of five interviews with each in which the conflicts were confronted failed to produce any significant change:

> The data of this study confirm students’ procedural, dynamic view of limit, that is, as an idealization of evaluating the function at points successively closer to a point of interest. The data also suggest that there are numerous idiosyncratic variations on this theme, some of them extremely difficult to dislodge. (Williams 1991, p.235)

4.2 Procedural consequences of conceptual difficulties

When faced with conceptual difficulties, the student must learn to cope. In previous elementary mathematics, this coping involves learning computational and manipulative skills to pass exams. If the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. At least this resonates with earlier experiences in arithmetic and algebra in which a sequence of manipulations are performed to get an answer.

The problem is that such routines very soon become just that—routine, so that student begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion. As a result, conceptual connections become less likely to be made. For instance, Eisenberg (1992) showed that students often failed to connect differentiation and integration as inverse processes, simply noting that there were distinct procedures to cope with each.

For those students who take an initial calculus course based on elementary procedures, there is evidence that this may have an unforeseen limiting effect on their attitudes when they take a more rigorous course at a later stage. Commenting on the results of a large study comparing the results of students taking advanced placement calculus courses in school, Ferrini-Mundy and Gaudard (1992) found that

> it is possible that procedural, technique-oriented secondary school courses in calculus may predispose students to attend more to the procedural aspects of the college course. (p.68)

Arriving at college and finding conceptual difficulties in the calculus, students can be seen to be developing short-term techniques for survival:

> Much of what our students have actually learned ...—more precisely, what they have invented for themselves—is a set of 'coping skills' for getting past the next assignment, the next quiz, the next exam. When their coping skills fail them, they invent new ones. The new ones don't have to be consistent with the old ones; the challenge is to guess right among the available options and not to get faked out by the teacher's tricky questions. … We see some of the 'best' students in the country; what makes them 'best' is that their coping skills have worked better than most for getting them past the various testing barriers by which we sort students. We can assure you that that does not necessarily mean our students have any real advantage in terms of understanding mathematics.
> (Smith & Moore 1991, p. 85)

Selden, Mason and Selden (1989; 1994) showed that students could learn to perform well on standard tasks, but as soon as a more unusual task was given, the success rate dropped alarmingly. This was investigated using two questionnaires. One had routine questions such as:

If $f(x)=x^{-1}$, find $f'(x)$,

or

If $f(x) = x^5+x$, where is f increasing?

The other consisted of non-routine questions such as:

Let $f(x)=\begin{cases} ax, & x \le 1, \\ bx^2+x+1, & x>1. \end{cases}$

Find a and b so that f is differentiable at 1.

Of nineteen grade A and B students, none could solve the non-routine problem above and two thirds could not complete *any* of five non-routine problems on the paper. Their average routine score was 74% but the average non-routine score was 20%.

4.3 Infinitesimal concepts

In an attempt to make the calculus more intuitively conceptual, one method might be to build on the intuitions that students adopt naturally when dealing with dynamic limit concepts. In his *Calculus made Easy*, at the turn of the century, Silvanus P. Thompson approached the calculus using tiny quantities and begged the reader 'not to give the author away nor to tell the mathematicians what a fool he really is'. It proved to be a satisfying book precisely because the notion of 'infinitely small' is so cognitively appealing. When a logical approach using infinitesimals was introduced as 'nonstandard analysis' (Robinson 1966) it had only limited impact. Its subsequent use in a calculus text (Keisler 1976) had a limited success, even though Sullivan (1976) showed that students following such an approach had a better grasp of the underlying concepts and were better on formal ε–δ questions than those following a standard calculus course.

Frid (1992; 1994) compared the effects of three approaches to the calculus: a technique-oriented calculus course, a 'concepts first' course, and one using intuitive non-standard analysis where the limit was described in terms of 'rounding off' values to give the limiting value. She found that students using infinitesimal language were far more likely to be able to verbalise subtle conceptual ideas. For instance, Jennifer (following the technique-oriented course) could easily handle a problem with implicit differentiation but could not explain what the limit notation for the derivative meant. On the other hand, Neill, following the non-standard course, explained the derivative saying 'if you were to magnify that function infinitely it would look like a straight line', then related it to the gradient where the 'rise and the run would be infinitesimal'. A closer look at the language used by the students will show a greater willingness to talk coherently about the concepts in infinitesimal terms. However, there are still underlying beliefs similar to those experienced by students approaching calculus in an intuitive dynamic way.

4.4 The underlying number system

Although mathematicians may think of the real numbers in terms of limits of decimal expansions, or a complete ordered field, or a corresponding geometric representation of the number line, closer inspection reveals our concept images to be considerably more diffuse and self-contradictory. For instance, we believe that a point has 'no size' and yet, somehow, that these can make up a visual 'real line'.

Romero i Chiesa and Azcárate-Gimenez (1994) asked students a number of conceptual questions about the real number line, both in terms of decimals and the visual representation. They found absolutely no evidence to suggest that students had any intuitive idea of the mathematical 'real line'. Three questions provoked interesting reactions:

- Imagine a number line. What do you see?
- Imagine this is magnified, what do you see now?
- What happens at infinite magnification?

47% of students questioned began by seeing the line as a whole and 28% saw elements in it—frequently reported as disks or as little spheres. At infinite magnification this changed to 20% seeing a line and 37% seeing individual elements. The response of the reader might be interesting on this one. The classical mathematician may say the real numbers have no infinitesimals, so the third part has no meaning, but using non-standard analysis it is perfectly in order to do the magnification in a larger field ***R****** and then restrict the view only to elements of the real line ***R***. The answer is that at infinite magnification, there will only be *one* real point in view (because two points in view would differ by an infinitesimal quantity before magnification, and in ***R***—which has *no* infinitesimals—these two points must be one and the same!)

Monaghan (1986) found that students were comfortable with whole numbers and rationals on the line, and came to be familiarised with other numbers such as $\sqrt{2}$, π and e, but regarded infinite decimals as somehow 'improper' because they 'go on forever' and never reach their final limit.

Using decimal representations has side-effects that are not always immediately apparent. Decimals expressed to a finite number of places are *discrete*, so that to, say, four decimal places, there is a *first* positive decimal, namely 0.0001. Wood (1992) found that a significant minority of mathematics majors after a year of analysis were able to affirm that there was no *least positive real number* (because if x were the least, $x/2$ would be smaller), but there was a *first positive number* ('point many noughts one'). This extrapolation of finite decimals can inadvertently cause different views with the limiting process as x tends to a depending on whether it is viewed geometrically (allowing 'smooth' movement) or as decimal numbers (perhaps in more discrete steps).

5. CALCULUS

5.1 Visualising Calculus Concepts

A potent visual approach using computer graphics is to magnify the graph of a function. This uses the same essential idea from non-standard analysis (that a differentiable graph under infinite magnification is a straight line). In the computer version, as the magnification increases, the graph looks less curved, and when it looks visibly straight, then the gradient of the graph is represented by the gradient of the line on the screen. Such an approach can use the visible limitations of computer graphics, emphasising that what is seen is only an approximation of the mental concept, yet making the notion of limit *implicit* in the magnification procedure rather than explicit in a formal definition.

By drawing graphs on the computer and having a second window in which part of the graph can be magnified, it is possible to see that some graphs look progressively less curved as they are magnified to a greater degree (Figure 12; Tall 1985).

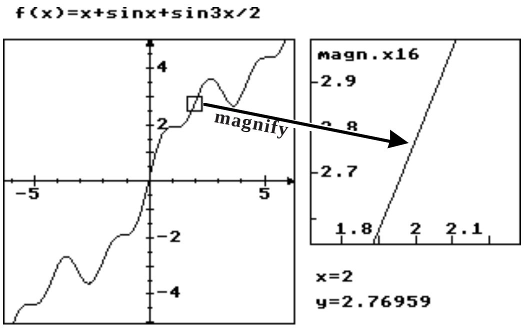


Figure 12: Magnifying a locally straight graph

A graph with this property is called 'locally straight'. It is possible to build up the gradient function of a locally straight graph by computing the numerical gradient between x and $x+c$ for small c at points along the graph, and hence, 'see' the gradient graph and experimentally conjecture its formula (figure 13).

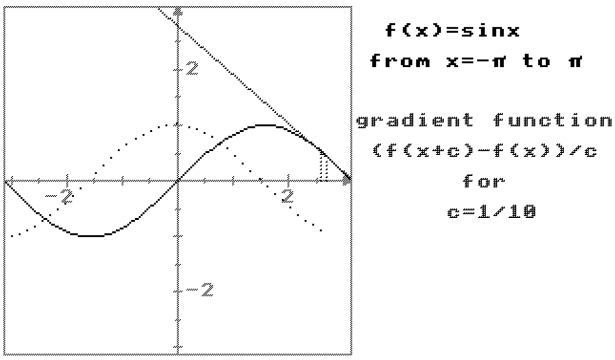


Figure 13 : Building the gradient function of sin*x*

It is also sensible to link this process with the symbolic formulae at the same time (particularly in simple cases such as $y=x^2$) so that the visual insight supports the symbols used for more sophisticated manipulations and computations. Such approaches have been adopted in syllabuses in the UK and elsewhere (School Mathematics Project 1991, Barnes 1991).

Further visual insights which support sophisticated ideas in analysis which were not long ago considered impossible to convey to novices now prove to be visually easy to imagine. For instance, the notion of different left and right gradients at a point can be seen magnified as a 'corner' and the graph of a non-differentiable function simply looks wrinkled at every level of magnification (Figure 14; Tall 1986).

Once differentiation is seen as the gradient of the graph under magnification, 'undoing' differentiation simply means knowing the gradient and finding the related graph. This generalises to finding the solutions of differential equations which can be performed numerically by computer software and displayed visually (figure 15; Hubbard & West 1990).

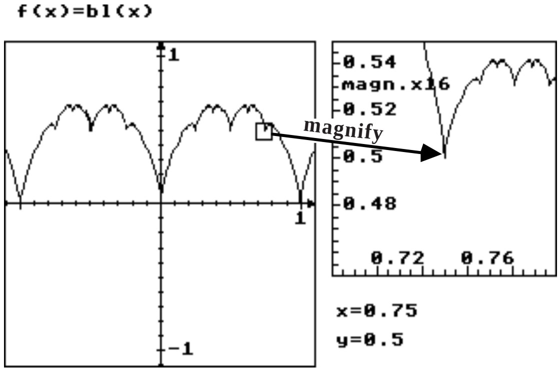


Figure 14: Magnifying the nowhere differentiable blancmange function

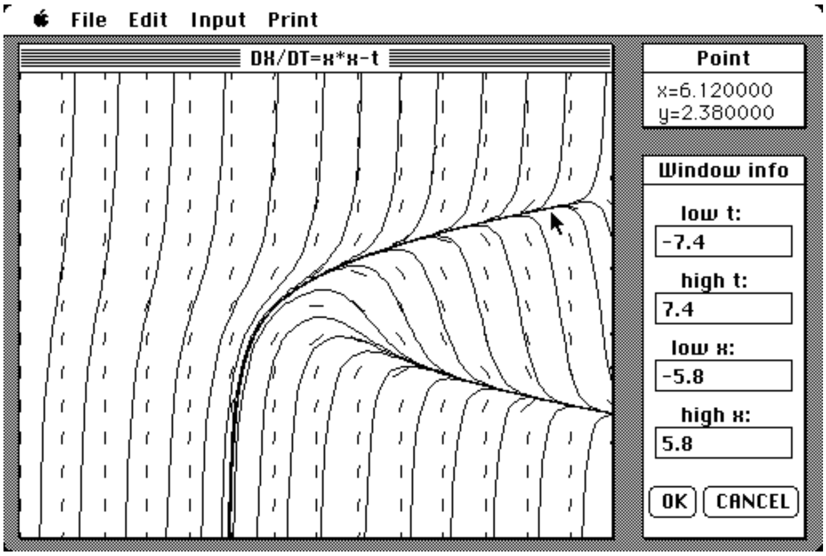


Figure 15: A family of solutions of a differential equation

Integration can be viewed as finding the area under a graph, and this can be visualised as adding up the approximate areas of thin strips under the graph. There are known cognitive difficulties here. For instance, Schneider (1993, pp.32, 33) reports that, in considering upper and lower sums for the function $y=x^3$ from 0 to 1 by taking more and more rectangles (figure 16), some students think that 'as long as the rectangles have a thickness, they do not fill up the surface under the curve, and when they become reduced to lines, their areas are equal to 0 and cannot be added.'

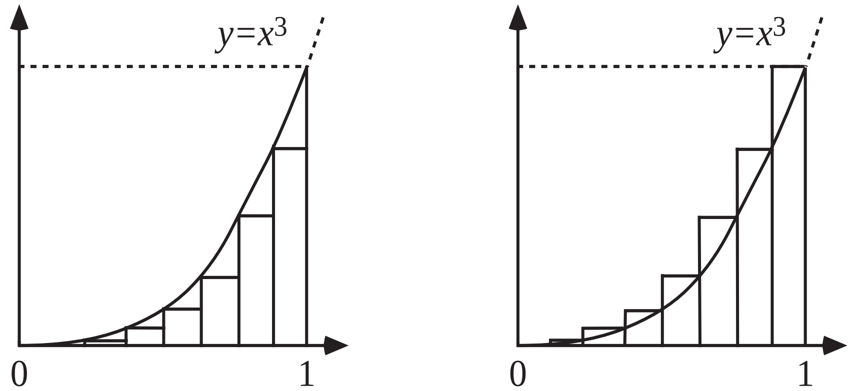


Figure 16: Lower and upper sums

The limit process contains implicit conceptual obstacles (Sierpiń ska 1985 1987). For instance some students believe that the process is potentially infinite, going on forever, but that it cannot reach its conclusion.

Given that the area $A(x)$ from a fixed point a to a variable point x to be considered as a function of x (which may prove difficult for students who conceive of functions purely in terms of formulae), the fundamental theorem of calculus says that $A'(x)=f(x)$. Visually the additional area under the curve from x to $x+h$ is $A(x+h)–A(x)$ (figure 17).

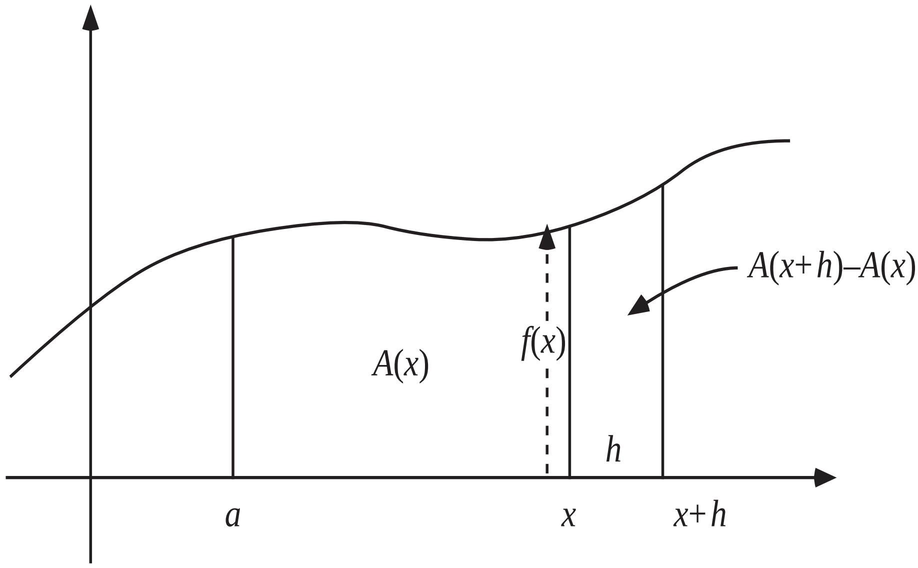


Figure 17: The area under a graph

Here there is only *one* strip to deal with and it may be more visually evident that, under appropriate conditions, as $h \to 0$, so

$$\frac{A(x+h)-A(x)}{h} \to f(x)$$

One insight is that a continuous graph ‘pulls flat’ when the vertical scale is kept constant and the horizontal is stretched, whilst looking though a fixed size viewing rectangle (figure 18; Tall 1986; 1991).

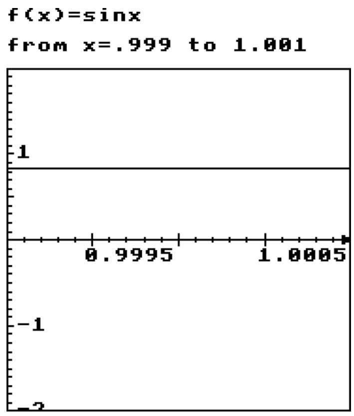


Figure 18: Horizontal stretching of the sine graph

Here the area from x to $x+h$ is seen to be approximately a rectangle height $f(x)$, width h and, as h tends to zero, the approximation may be imagined to ‘get better’, so that $(A(x+h)–A(x))/h$ more closely approximates $f(x)$, as required. Again there are clear cognitive obstacles, for instance in trying to imagine how an approximation becomes an equality in the limit.

The ‘pulling flat’ property can be seen to be equivalent to point-wise continuity by imagining the pixel to represent a height $f(x)\pm\varepsilon$, then, if it pulls flat in the window, there must exist a $\delta>0$ such that picture of the graph from $x–\delta$ to $x+\delta$ lies within the pixel height, $f(x)\pm\varepsilon$. That is,

given $\varepsilon>0$, there is a $\delta>0$ such that
$x-\delta < t < x+\delta$ implies $f(x)-\varepsilon < f(t) < f(x)-\varepsilon$.

There seems to be little published research on the formal beginnings of analysis, however, my own personal experience suggests that above average (as opposed to gifted) students can learn to discuss the concepts meaningfully based on the visual imagery, but that the translation to the formal proof proves difficult. First the student usually imagines the definition to *describe* an existing object, rather than *define* the object by deducing its properties (Sierpiń ska 1992), thus finding it strange to 'prove' obvious properties that seem already to be true. Then there are further difficulties because of the complex use of quantifiers and the formality of the deductions. The decision of most UK universities to abandon the teaching of formal analysis as a first year university course is evidence of its huge cognitive difficulty.

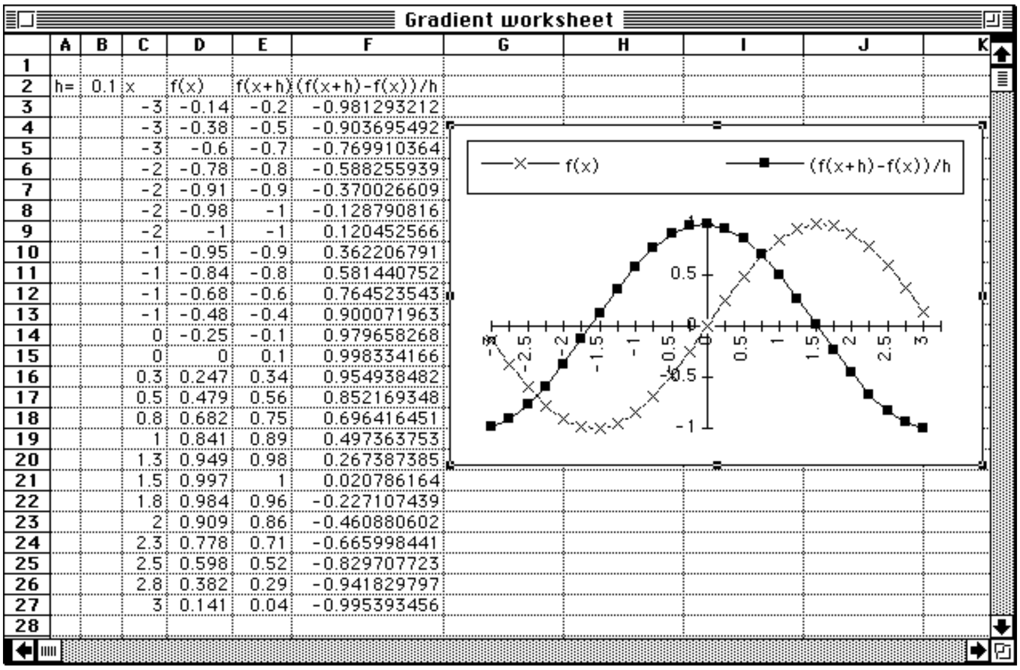
5.2 Numerical Representations

Numerical representations can occur in a number of ways, including:

- using software which prints out tables of values, possibly as part of a 'multiple-representation package',
- using spreadsheets to build up tables of values (which may then be represented graphically),
- student programming of numerical routines.

'Tables of values' are a favourite device of mathematics educators which hardly figured in traditional calculus. They offer a simple numeric representation to complement the visual graph and the symbolic formula. However, there are two very distinct uses for tables of values. One is as a genuine table of data, for instance, from experimental readings in a real-world context. Here the 'change' from one data reading to the next is concerned with the discrete theory of finite differences rather than the limit theory of the calculus. The other use is to print out a selected table of evaluations from a function given by a procedure or formula. Here new tables can be generated as required to do such things as search for a change in sign and home in on a zero of the function.

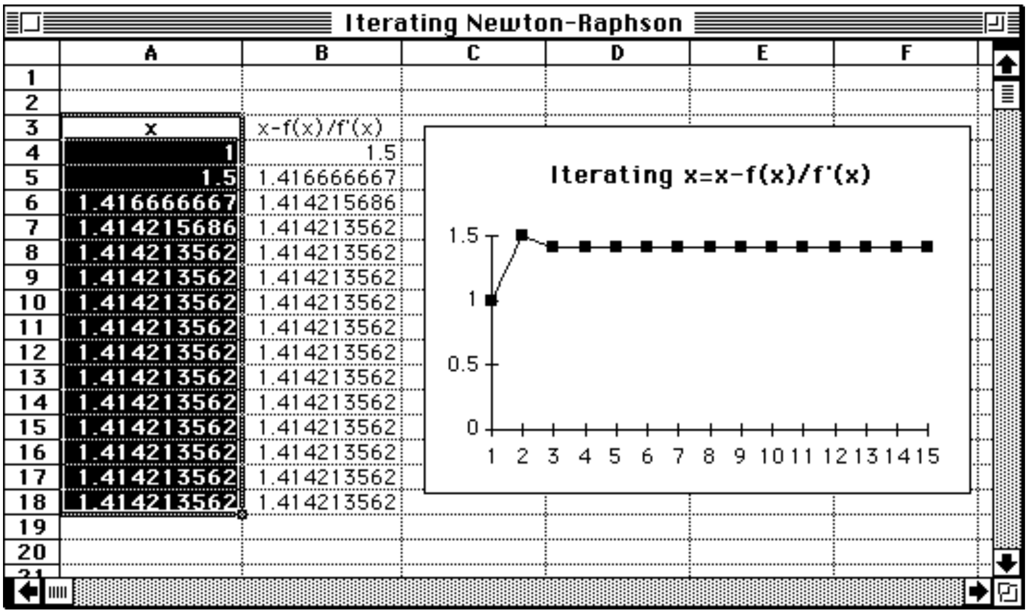
Placing the data (either from an experiment, or from a function formula) into a spreadsheet gives opportunities for investigative exploration (and subsequent graphic interpretation). For instance, it is possible to design a worksheet to draw a graph and its gradient (figure 19).



	A	B	C	D	E	F
1						
2	h=	0.1	x	f(x)	f(x+h)	(f(x+h)-f(x))/h
3			-3	-0.14	-0.2	-0.981293212
4			-3	-0.38	-0.5	-0.903695492
5			-3	-0.6	-0.7	-0.769910364
6			-2	-0.78	-0.8	-0.588255939
7			-2	-0.91	-0.9	-0.370026609
8			-2	-0.98	-1	-0.128790816
9			-2	-1	-1	0.120452566
10			-1	-0.95	-0.9	0.362206791
11			-1	-0.84	-0.8	0.581440752
12			-1	-0.68	-0.6	0.764523543
13			-1	-0.48	-0.4	0.900071963
14			0	-0.25	-0.1	0.979658268
15			0	0	0.1	0.998334166
16			0.3	0.247	0.34	0.954938482
17			0.5	0.479	0.56	0.852169348
18			0.8	0.682	0.75	0.696416451
19			1	0.841	0.89	0.497363753
20			1.3	0.949	0.98	0.267387385
21			1.5	0.997	1	0.020786164
22			1.8	0.984	0.96	-0.227107439
23			2	0.909	0.86	-0.460880602
24			2.3	0.778	0.71	-0.665998441
25			2.5	0.598	0.52	-0.829707723
26			2.8	0.382	0.29	-0.941829797
27			3	0.141	0.04	-0.995393456
28						

Figure 19: Using a spreadsheet to calculate and display the graph and gradient of sin*x*

Spreadsheets are also particularly good for iterative work, for example using a recurrence formula to calculate the limit of a sequence such as the Newton-Raphson method of finding the zero of a differentiable function (figure 20). Hunter et al. (1993) found this to be a more effective method of showing the workings of the iteration in successive lines of code as opposed to a more enigmatic use of recursion with a symbol manipulator.



	A	B
3	x	x-f(x)/f'(x)
4	1	1.5
5	1.5	1.416666667
6	1.416666667	1.414215686
7	1.414215686	1.414213562
8	1.414213562	1.414213562
9	1.414213562	1.414213562
10	1.414213562	1.414213562
11	1.414213562	1.414213562
12	1.414213562	1.414213562
13	1.414213562	1.414213562
14	1.414213562	1.414213562
15	1.414213562	1.414213562
16	1.414213562	1.414213562
17	1.414213562	1.414213562
18	1.414213562	1.414213562

Figure 20: Using a spreadsheet for Newton-Raphson approximations to a root of $f(x)=x^2-2$

Spreadsheets can be used in other ways for imaginative display of data. Abramovich (1995) used them to study the cauchy convergence of sequences by calculating the difference between s_m and s_n in the (m, n)th position, placing different symbols according as to whether it was bigger or less than a given positive ε. The resulting display allows the viewer to select a value of N such that, when m, n exceed N, then $|s_m - s_n| < \varepsilon$.

Programming numerical procedures to carry out mathematical processes can be of value in calculus. For instance, programming iterative solutions of equations ('undoing' function problems) or programming the solutions of differential equations ('undoing' differentiation problems) or programming the area under a curve ('doing' integration). Such approaches enable the student to investigate ideas experimentally, for example, to see what kind of errors occur with different methods of calculating areas (such as 'first ordinate', 'last ordinate', 'mid-ordinate', 'trapezium rule' and 'Simpson's rule'). However, this involves two separate skills—programming and conceptualising mathematical concepts—and the two may prove difficult to do at the same time for students of average ability and even above. Cowell and Prosser (1991) report a mixture of 'good and bad news' in the use of True BASIC.

> The students largely agreed that the computer assignments were well integrated with the rest of the course, and that learning the necessary programming was easy, but they disagreed that the computer enhanced their interest in the course material, they disagreed that the computer should be dropped and they were divided on whether the computer assignments were a valuable part of the course (Cowell & Prosser 1991, pp.152, 153)

Li and Tall (1993) report that programming functions in structured BBC BASIC helped students conceive of functions as mental objects and to conceptualise sequences and series as functions, but did not help greatly in thinking of the limit as a concept rather than a process. On the contrary, the sequences programmed (including a sequence s_n taking the value $1/n^2$ if n is prime, $1/n^3$ if not prime and even, $1/n!$ otherwise) sometimes took considerable time to stabilise to 8 significant figures and so gave the impression that the limit may not be reached. Some students sensed that an increasing sequence bounded above might not be convergent. This led to an open discussion on the completeness property which remained unresolved because students who were unsure if a sequence converged were unwilling to accept that it could be deemed to be convergent merely by asserting a 'completeness axiom'. These students were 'capable' rather than 'gifted' and when they did an analysis course two years later with another lecturer, only a small minority could shake off inappropriate images to use the definitions of limits to prove that a function was continuous or differentiable 'from first principles' (Pinto & Gray 1995).

5.3 Conceptual Programming

Programming in calculus courses often involves numerical algorithms, sometimes in the hope that this will give support to later conceptual ideas, some of which prove not to work nearly as easily as might be hoped. On the other hand, the computer language ISETL (Interactive SET Language) is designed specifically to mirror mathematical ideas, such as the definitions of sets, with functions as sets of ordered pairs or processes of assignment, whose names can be used as the input to other functions. Thus it is possible to defined a function *D* which takes a function *f* as input and return a function *D*(*f*) which is a numerical derivative of *f*. The following code in ISETL produces the numerical derivative of *f* for *h*=0.0001:

```
D : = func(f);
  return func(x);
        return((f(x+0.0001)–f(x))/0.0001;
  end;
end;
```

where **return func(x)** denotes that **D** returns a function of **x** using the given formula. For any function whatever, such as the exponential function **exp**, then **D(exp)** will be another function. If **x** is a number, then **D(exp)(x)** is a number, namely the value of the numerical derivative of **exp** calculated at **x**.

A mid-ordinate approximation to the Riemann integral from *a* to *b* with *n* steps can be programmed as:

```
RiemLeft : = func(f,a,b,n);
  x := [a + ((b–a)/n)*i : i in [0..n]];
  return %+[f(x(i–1))*(x((i)–x(i–1)) : i in [1..n]];
end;
```

(where the symbol **%+** stands for the summation symbol Σ). The procedure computes the equally spaced set of values **x(0)**, ..., **x(n)**, then calculates the Rieman sum of the areas of strips width **x((i)–x(i–1)**, height **f(x(i–1))** as **i** varies from **1** to **n**.

The approximate integral operator can then be defined for any function *f* as

```
Int : = func(f,a);
  return func(x);
        return( RiemLeft(f,a,x,25);
  end;
end;
```

(To improve accuracy, instead of using 25 steps in the function, it is possible to repeat the computation for an increasing number of strips and only return the value when it has stabilised to an appropriate accuracy.) The students are encouraged to combine **D** and **Int**, such as **Int(D(f),a)** and **D(Int(f,a))** in various activities in an effort to help them construct the relationship between differentiation and integration (Dubinsky & Schwingendorf 1991). This method still inhabits the world of numerical approximations and the consequent inherent cognitive complications involving numbers and limits, but it has the advantage of constructive activities designed to mirror the corresponding mathematical ideas.

5.4 Computer Algebra Systems

Computer algebra systems (or symbol manipulators) are now being used more extensively in teaching calculus, from courses based on software notebooks that include symbol manipulation and graph-drawing in *Mathematica* (Brown, Porta & Uhl 1990; 1991a), to laboratory workshops added to standard courses in *Maple* (e.g. Muller 1991), and research projects (e.g. Heid 1988; Palmiter 1991).

Brown, Porta and Uhl; (1990; 1991b) report sophisticated student usage of symbolic facilities provided in notebooks in *Mathematica*, with students passing the symbol manipulation to the software whilst concentrating on other aspects of the problem. Muller (1991) reports a project in which a first course (1988) was received enthusiastically by volunteers and was followed by two successive compulsory courses (1989; 1990) which still showed some gains, though at a more realistic level. An important factor in this project was a significant reduction in student withdrawal rates and failure rates.

Heid (1988) used graphical software to illustrate concepts and the early computer algebra system *MuMath* to carry out symbol manipulation, only practising paper and pencil skills in the last three weeks of a fifteen week course. The students performed better on conceptual questions and were statistically not significantly different from control students doing a full fifteen week course on standard techniques.

Palmiter (1991) used the symbolic software MACSYMA to teach one cohort of students a first course in integration for five weeks whilst a parallel cohort studied a traditional course for a full ten weeks. The MACSYMA students used the software to carry out routine computations whilst the traditional students were taught the techniques. Both groups took a conceptual examination and a computational examination at the end. The conceptual examination was taken by both groups under identical conditions, the experimental students were allowed to use MACSYMA in the computational examination but had only one hour whilst the control students were given two hours. The results showed in conceptual questions the experimental students achieving an average of 89.8% (±15.9) compared with the average traditional course score of 72.0% (±20.4). and on computational questions an average of 90.0% (±13.2) compared with 69.6% (±24.2).

This gives clear indications that a ‘student plus manipulation tool’ can be more successful in conceptual and computational tasks than a student working in a traditional manner.

However, other experiments do not always show significant improvements in performance, particularly in paper and pencil manipulative skills. Comparing students in a computer laboratory using *Derive* and a traditional course, Coulombe and Mathews (1995) found no significant differences in knowledge, paper and pencil manipulation, conceptual understanding, or higher order thinking skills, although it produced similar levels of performance whilst giving students additional familiarity with computer technology.

The use of software with graphical facilities and symbol-manipulation changes students conceptions of the calculus and their abilities to carry out the related skills. For instance, having graphs drawn by technology does not involve explicitly calculating and plotting function values. Hunter et al. (1993) found that a third of the students in one class could answer the following question before the course, but not after:

'What can you say about u if $u=v+3$, and $v=1$?'

During the course they had no practice in substituting values into expressions and the skill seems to have receded until it is not used in the post-test.

By the same token, Monaghan et al. (1994) found that some students using a computer algebra system to carry out the process of differentiation responded to a request for an explanation of differentiation by describing the sequence of key-strokes that were necessary to get the result. It appears that some students may simply replace one procedure which has little conceptual meaning with another.

Changes in learning are caused by a variety of factors of which the technology is only one. Coston (1995) studied the effects on grades of cooperative learning, with and without the use of technology. The results showed no significant differences using technology alone but cooperative learning plus technology produced a significant improvement in attitude whilst cooperative learning by itself produced a highly significant improvement in problem solving.

In a technological age where grocery bills are totalled at the checkout by computer technology and computers are used throughout business and commerce, the need to test a student in the absence of the technology may become increasingly questioned. Yet while some 'conceptually oriented' courses have shown students able to respond well to conceptual questions, able to perform manipulations better using the technology and performing no worse at paper and pencil skills with a little practice, the knowledge being obtained is certainly different and is likely to have new strengths and also hidden flaws.

6. The Future for Functions and Calculus

In the mid-1980s, calculus was under attack from discrete mathematics, which seemed to some to be the mathematics of the computer and so the mathematics of the future. At the time I wrote an article entitled 'W(h)ither calculus', with an intended pun on the first word with or without its 'h', for there were suggestions that the calculus would wither away and die (Tall 1987). Subsequent events have shown the reverse is true. Calculus is at the forefront of curriculum reform in mathematics with its vigour renewed by the advent of the computer. Mathematicians are discovering their zest for experiment and adventure and are passing on their enthusiasms to their students. At the same time, the notion of function has been seen as a central theoretical construct. The first waves of reform have stimulated the system and have been accompanied by clarion calls declaring the new dawn. The time for evaluation and cool consideration has arrived but should not be allowed to dampen the ardour which many of the reformers have passed on to their students.

Just as enthusiasms for the theory of 'new mathematics' in the 1960s had to be tempered by the realities of the growth of knowledge in the individual, so the fundamental nature of the function concept is beginning to be seen in a more realistic light of cognitive development. It continues to be viewed as a fundamental mathematical notion and has a prominent role in the curriculum, but the underlying cognitive conceptual difficulties are beginning to be better understood, even if it is proving more problematic to cater for them.

Calculus has broadened in its meaning from traditional symbolic techniques to a wider science of how things change, the rate at which they change, and how their growth accumulates. Instead of being only an intellectual challenge for the elite, it has widened its appeal to allow experimental exploration and a quest for meaning without losing sight of the long-term need for meaningful proof. It exists in a variety of forms that allow students to harness the power of computer software to seek insight from a variety of viewpoints.

The discovery of the calculus over three centuries ago was one of the most significant events in the evolution of civilization. The momentous changes occurring with the growth of information technology in the last decade show calculus still playing its central role.

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