The Mathematical Side of M. C. Escher

Doris Schattschneider

While the mathematical side of Dutch graphic artist M. C. Escher (1898–1972) is often acknowledged, few of his admirers are aware of the mathematical depth of his work. Probably not since the Renaissance has an artist engaged in mathematics to the extent that Escher did, with the sole purpose of understanding mathematical ideas in order to employ them in his art. Escher consulted mathematical publications and interacted with mathematicians. He used mathematics (especially geometry) in creating many of his drawings and prints. Several of his prints celebrate mathematical forms. Many prints provide visual metaphors for abstract mathematical concepts; in particular, Escher was obsessed with the depiction of infinity. His work has sparked investigations by scientists and mathematicians. But most surprising of all, for several years Escher carried out his own mathematical research, some of which anticipated later discoveries by mathematicians.

And yet with all this, Escher steadfastly denied any ability to understand or do mathematics. His son George explains:

> Father had difficulty comprehending that the working of his mind was akin to that of a mathematician. He greatly enjoyed the interest in his work by mathematicians and scientists, who readily understood him as he spoke, in his pictures, a common language. Unfortunately, the specialized language of mathematics hid from him the fact that mathematicians were struggling with the same concepts as he was. Scientists, mathematicians and M. C. Escher approach some of their work in similar fashion. They select by intuition and experience a likely-looking set of rules which defines permissible events inside an abstract world. Then they proceed to explore in detail the consequences of applying these rules. If well chosen, the rules lead to exciting discoveries, theoretical developments and much rewarding work. [18, p. 4]

In Escher's mind, mathematics was what he encountered in schoolwork—symbols, formulas, and textbook problems to solve using prescribed techniques. It didn't occur to him that formulating his own questions and trying to answer them in his own way was doing mathematics.

Until 1937

M. C. Escher grew up in Arnhem, Holland, the youngest in a family of five boys. His father was a civil engineer and his four older brothers all became scientists. The home atmosphere may have instilled in him some habits of scientific inquiry, including the patient, methodical approach that would characterize his later work. Also, the young boys were given regular lessons in woodworking techniques that would later become very useful to Escher in making woodcuts.

His school life may have been less useful than his home life. Recalling his school years, Escher once confessed "I was an extremely poor pupil in arithmetic and algebra, and I still have great difficulty with the abstractions of figures and letters. I was slightly better at solid geometry because it appealed to my imagination, but even in that subject I never excelled at school" [1, p. 15]. He did well in drawing, however, and his high school art teacher encouraged him to make linocuts.

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In 1919 Escher entered the Haarlem School for Architecture and Decorative Arts intending to study architecture, but with the advice of his drawing and graphic arts teacher, Samuel Jessurun de Mesquita, and the consent of his parents, soon switched to a program in graphic arts. Among his prints executed while in Haarlem are three that show plane-filling; two of these are based on filling rhombuses, and one has a rectangle filled with eight different elegant heads, four upside down, each repeated four times [53, pp. 7–8]. Plane-filling would soon become an obsession.

Upon finishing his studies at the Haarlem School in 1922, he traveled for most of a year throughout Italy and Spain, filling a portfolio with sketches of landscapes and details of buildings, as well as meticulous drawings of plants and tiny creatures in nature. During this odyssey, he visited the Alhambra in Granada, Spain, and there marveled at the wealth of decoration in majolica tiles, sketching a section that especially attracted him "for its great complexity and geometric artistry" [1, pp. 24, 41]. This first encounter with the tilings in the Alhambra likely increased his interest in making his own tilings. In any case, during the mid-1920s, he produced a few periodic "mosaics" with a single shape, some of them hand-printed on silk [53, p. 11]. Unlike the Moorish tiles that always had geometric shapes, Escher's tile shapes (which he called "motifs") had to be recognizable (in outline) as creatures, even if only of the imagination. These early attempts show that he understood (intuitively, at least) how to utilize basic congruence-preserving transformations—translations, half-turns (180° rotations), reflections and glide-reflections—to produce his tilings.

Escher married in 1924, and the couple settled in Rome, where two sons were born. Until 1935 he continued to make frequent sketching trips, most in southern Italy, returning to his Rome studio to compose his sketches for woodcuts and lithographs. In 1935, with the growing rise of Fascism in Italy and his sons in ill health, Escher felt it best to move his family from Italy to Switzerland. In 1936 he undertook a long sea journey, and during the trip he spent three days at the Alhambra, joined by his wife Jetta. There they made careful color sketches of many of the majolica tilings. This second Alhambra visit, coupled with his move from the scenery of Italy, marked an enormous change in his work: landscapes would be replaced by "mindscapes".[1] No longer would his sketches and prints be inspired by what he found in mountainous villages, nature, and architecture. Now the ideas would be found only in the recesses of his mind.

Escher later wrote that after this Alhambra visit, "I spent a large part of my time puzzling with animal shapes" [1, p. 55]. By carefully studying the Alhambra sketches and noting the geometric relationships of the tiles to one another, he was able to make a dozen new symmetry drawings of interlocked motifs.[2] One of these showed interlocked Chinese boys. In spring 1937 he produced his first print that used a portion of a plane-filling to produce a metamorphosis of figures. In *Metamorphosis I*,[3] the buildings of the coastal town of Atrani morph into cubes which in turn evolve into the Chinese boys [53, pp. 19, 286]. The print was a fantasy, linking his new interest in plane-fillings with his love of the Amalfi coast, but Escher never liked it because it didn't tell a story—how do you link Chinese boys to an Italian town?

In July 1937 the Escher family moved to a suburb of Brussels, where a third son was born. That October Escher showed his meager portfolio of symmetry drawings to his older half-brother Beer, a professor of geology, who immediately recognized that these periodic patterns would be of interest to crystallographers, since crystals were defined by their periodic molecular structure. He offered to send Escher a list of technical papers that might be helpful. There were ten articles in Beer's list, all from *Zeitschrift für Kristallographie*, published between 1911 and 1933, by F. Haag, G. Pólya, P. Niggli, F. Laves, and H. Heesch [53, pp. 24, 337]. Escher found only the articles by Haag and Pólya useful.

Haag's article [28] provided a clear definition for Escher of "regular" plane-fillings and also provided some illustrations. In one of his copybooks, Escher carefully wrote Haag's definition of "regular division of the plane" (here translated):

> *Regular divisions of the plane consist of congruent convex polygons joined together; the arrangement by which the polygons adjoin each other is the same throughout.*

In the same copybook, Escher also sketched several of Haag's polygon tilings. After studying them, he quickly discovered that the word "convex" in Haag's definition was superfluous, for by manipulating the tile's shape, he was able to sketch several examples of nonconvex polygonal tilings. It was probably at this point that he inserted parentheses around the word "convex" in Haag's definition. Of course he also readily discovered that the word "polygon" was far too restrictive for his purposes; it could easily

[1] *The title of a 1995 exhibit of Escher's work at the National Gallery of Canada in Ottawa was titled "M. C. Escher: Landscapes to Mindscapes".*

[2] *Escher's colored plane-fillings have been called tessellations, periodic drawings, tilings, and symmetry drawings. I prefer to use the last term.*

[3] *All of Escher's prints that are named in this essay can be found in the catalogs* [1] *and* [37], *and many of them can also be found in* [20], [59], *and on the official website* `www.mcescher.com`.

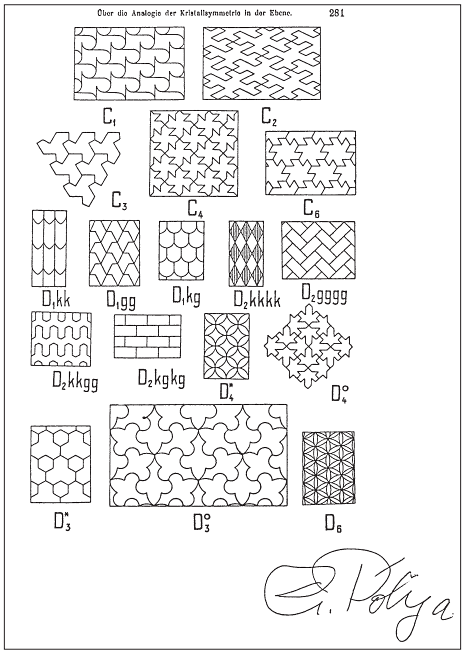

Figure 1. A copy of the display in [43], signed by Pólya.

be replaced by "tile" or "shape". Haag's definition (with Escher's amendments) was adopted by Escher and would guide all of his symmetry investigations. He later carefully recorded the definition on the back of his symmetry drawing 25 (1939) of lizards (the drawing is depicted in Escher's lithograph *Reptiles*).

Pólya's article [43] would have a great influence on Escher. Escher carefully copied (by hand, in ink) the full text that outlined the four isometries of the plane and announced Pólya's classification of periodic planar tilings by their symmetry groups. Pólya was evidently unaware that this classification had been carried out by Fedorov more than thirty years earlier. Escher was already intuitively aware of the congruence-preserving transformations Pólya spoke of but probably didn't understand any of the discussion about symmetry groups. What struck him was Pólya's full-page chart that displayed an illustrative tiling for each of the seventeen plane symmetry groups (Figure 1). Escher carefully sketched each of these seventeen tilings in a copybook and studied them, map-coloring some of them [47]. Among these, there were tilings that displayed symmetries he had not recorded in the Alhambra; for example, tilings whose only symmetries other than translations were glide-reflections or fourfold (90°) and twofold (180°) rotations. Within one month of studying these, Escher had completed his first symmetry drawings displaying fourfold rotation symmetry: squirming lizards interlocked four at a time, pinwheeling where four feet met [53, p. 127]. He featured a portion of one of these drawings at the center of his woodcut *Development I*, completed in the same month.

Escher was so grateful for the help that Pólya's paper provided that he wrote to the mathematician to thank him. He sent Pólya the print *Development I* and asked the mathematician whether or not he had written a book on symmetry for "laymen" as his article indicated he had hoped to do. Although a writer once characterized Pólya's reply as polite but formal, indicating he hadn't written the hoped-for book [53, p. 22], Pólya wrote to me in 1977 that he and Escher had corresponded more than once and that he regretted losing the correspondence in his haste to come to America in 1940. A recent discovery of a forgotten suitcase full of Pólya's notes and other collected letters and papers, now in the Pólya archives at Stanford University, shows that Pólya even sent Escher his own attempt at an Escher-like tiling. Among these papers is Pólya's drawing of a tiling by snakes, inscribed "sent to MCE", at the address where Escher resided from 1937 to 1940. Also, there is an outline of Pólya's never-completed book *The Symmetry of Ornament* and many sketches of tilings, both for the planned book and for the 1924 article that so influenced Escher [53, pp. 335–36].

Escher as a Mathematical Researcher

From 1937 to 1941 Escher plunged into a methodical investigation that can only be termed mathematical research. Haag's article had given him a definition of "regular division of the plane", and Pólya's article showed him that there were many tile shapes that could produce these. He wanted to find more and characterize them. The questions he pursued, using his own techniques, were:

(1) *What are the possible shapes for a tile that can produce a regular division of the plane, that is, a tile that can fill the plane with its congruent images such that every tile is surrounded in the same manner?*

(2) *Moreover, in what ways are the edges of such a tile related to each other by isometries?*

The only isometries that Escher allowed in order to map a tile to an adjacent tile were translations, rotations, and glide-reflections—a reflection would require a tile's edge to be a straight segment, not a natural condition for his creature tiles. In 1941–1942 he recorded his many findings in a definitive *Notebook* that was to be his private encyclopedia about regular divisions of the plane and how to produce and color them [46], [48], [53]. The *Notebook* had two parts: its cover inscribed (here translated) "Regular divisions of the plane into asymmetric congruent polygons; I Quadrilateral systems MCE 1-1941 Ukkel; II Triangle systems X-1942, Baarn".

Escher's study of "quadrilateral systems" was extensive. He represented these tilings symbolically with a grid of congruent parallelograms in which each parallelogram represented a single tile. He shaded the grids checkerboard style, so that each parallelogram shared edges only with parallelograms of the opposite color. He was interested in asymmetric tiles (after all, his creature tiles were to be primarily asymmetric), and in order to indicate the asymmetry, placed a hook inside each parallelogram. The hook provided orientation, while small circles and squares on the tile's boundary indicated twofold and fourfold centers about which the tile could rotate into an adjacent tile. Escher was aware that certain symmetries required special parallelogram grids and so considered five different categories: arbitrary parallelogram,

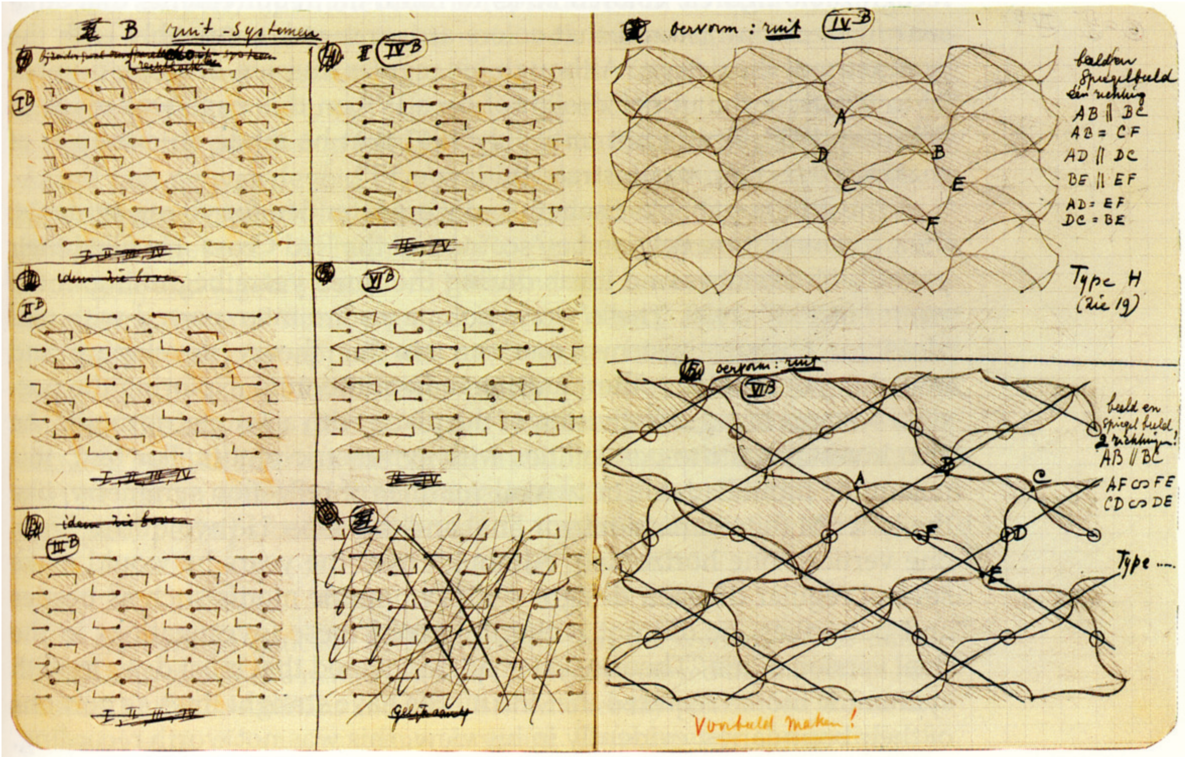

Figure 2. A copybook page showing Escher's method of investigation of regular divisions of the plane. His symbolic notation is explained in our text.

rhombus, rectangle, square, and isosceles right triangle (a grid of squares in which the diagonals have been drawn). He labeled these A–E, respectively. As he sought to answer his two questions, he filled the pages of several school copybooks with his sketches of marked grids representing tilings, scratching out those that didn't work out or that duplicated an earlier discovery. Each time he discovered a marked grid that represented a regular division of the plane, he recorded it and made an example of a tiling with a "shaped tile", its vertices marked by letters.

To quickly record how each edge of a tile was related to another edge of the same tile or an adjacent tile, Escher devised his own notation: = meant "related by a translation" and || "related by a glide-reflection". An S on its side meant "related by a 180° rotation" and L meant "related by a 90° rotation". Figure 2 shows one copybook page with five different "rhombus systems" on the left and shaped tilings for two of these systems on the right. Note Escher's "voorbeeld maken!" at the bottom of the page—"make an example!" His results were recorded entirely visually, with no need for words. Ultimately he found ten different classes of these tilings and numbered the classes I – X. His *Notebook* charts giving both visual and descriptive versions of the classes are in [53, pp. 58–61].

To discover still other regular divisions, those for which three colors would be required for map-coloring, Escher employed a technique that he called "transition". Figure 3 recreates one of his examples. He would begin with a two-color regular division from one of his ten categories (Figure 3 begins with type II^A). Each of these categories had four tiles meeting at every vertex and required only two colors. He would then choose a tile and a segment of the boundary that connected one of its vertices (say B) to another carefully chosen boundary point (say A) that was not a vertex of the tiling (Figure 3a). Using A as a pivot point, he would then pivot the boundary segment connecting A and B (stretching it if necessary) so that vertex B slid along the boundary of the tile, stopping at a new position (say C). Repeating this on the corresponding segments of the boundaries of all tiles produced a new tiling with vertices at which three tiles met, requiring three colors for map-coloring (Figure 3b). The process could be continued with the new segment AC, sliding C along the boundary until it reached a vertex D of the original tiling. This produced a new tiling that again required only two colors (Figure 3c). At the intermediate (3-color) stage, the network of tile edges was certainly not homeomorphic to the original, but surprisingly, at the end (2-color) stage, the new network of tile edges might also not be homeomorphic to that of the original tiling. Escher thought of the intermediate (3-color) tiling as having components of both the beginning and ending 2-color tilings, and so labeled it with both types. In Figure 3, his type II^A system is transformed to II^A–III^A, and that is transformed to system III^A. In this instance, the tiles in the final tiling have three, not four, edges

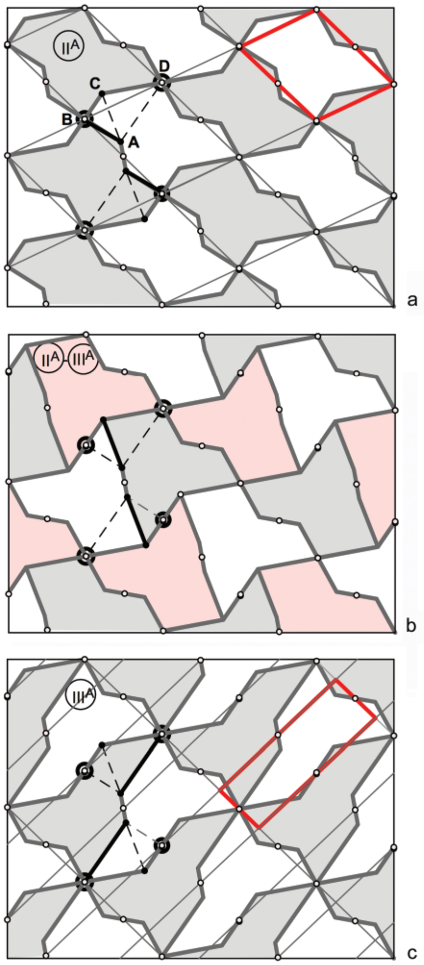

Figure 3. Escher's transition process takes (a) a 2-colored tiling with 180° rotation centers at tile vertices and midpoints of edges, to (b) a 3-colored tiling, to (c) a 2-colored tiling with 180° rotation centers at midpoints of tile edges.

and meet six at a vertex; Escher noted that this was an exceptional case [53, p. 62].

Escher did not record these discoveries with words, but in his *Notebook* he displayed sixteen pages of carefully drawn illustrations of transitions that cover all of his ten categories [53, pp. 62–69]. In many cases, he discovered more than one distinct transition of the same tiling. Using today's terminology, he discovered how to produce tilings of different isohedral types beginning with a single isohedral tiling. And he also recorded in a chart (a digraph!) which of his ten categories led to others. This chart makes clear that his process of transition can change the topological and combinatorial properties of a tiling but not change its symmetry group [48], [53, p. 60].

The last section of Escher's "quadrilateral systems" study summarizes in ten pages his investigations of what he called "2-motif" tilings [53, pp. 70–76]. He would begin with a regular division, map-colored with two or three colors, then split each tile (in the same manner) into two distinct shapes, so that the resulting tiling could be colored with two colors. This investigation was spurred by his fascination with what he called "duality"— many of his prints play with the idea of interchanging the role of figure and ground, or juxtaposition of opposites. *Sky and Water I* and *Circle Limit IV* (*Angels and Devils*) are famous examples.

For example, in *Sky and Water I* (Figure 4) a horizontal row of interlocked flat tiles at the center alternates white fish and black birds, dividing the print into upper and lower halves (sky and water). The fish in this center row can serve as figure and the birds as ground, or vice versa. But as the eye moves upward from this row of tiles, the creatures separate and take on distinct roles. The black birds become three-dimensional as they rise, while the white fish melt to become sky. The fish become the background against which figures of birds fly. As the eye moves downward from the center row of tiles, the opposite transformation takes place. Now the fish gain three-dimensional form and the black birds dissolve to become water in which the fish swim. In mathematics, the essence of dual objects is that each completely defines the other, such as a set and its complement, a statement and its negation. In addition to the figure/ground duality, there are other kinds of duality represented in this single print: black and white, sky and water. And opposites: bird and fish often denote opposites (think "neither fish nor fowl"), and in the print, each bird is placed exactly opposite a fish, with the invisible surface of the water acting as a compositional mirror.

Part II of Escher's *Notebook* is brief, devoted to what he called "triangle systems"—regular divisions having 120° rotation centers (system A) or 60°, 120°, and 180° rotation centers (system B). After explaining the necessary placement of rotation centers, he records only twenty different tilings, several with two motifs, and all carefully map-colored to respect symmetry. Most require three colors. Unlike in his quadrilateral systems, some of his tiles have rotation symmetry, and from these he derives other tiles with one or two motifs [53, pp. 79–81].

In 1941, as he was nearing the end of these investigations, Escher and his family moved to Baarn, Holland, where he would spend all but the last two years of his life. In the years following, he produced more than 100 regular divisions of the plane, each final version numbered and carefully drawn on graph paper, its creature tiles outlined in ink, map-colored using watercolors, respecting the symmetries of the tiling. As his portfolio of symmetry drawings grew, he referred to it as his "storehouse". Fragments of these drawings would be featured in many prints, notecards, exhibit announcements, painted and tiled public works, and even carved on the surface of a ball. In all, there are 134 numbered symmetry drawings and many unnumbered sketches.

Escher carried out several other minor mathematical investigations in order to achieve certain effects in his art. Some of these results were recorded in a notebook entitled *Regular Division of the Plane: Abstract Motifs, Geometric Problems*, and others were gathered in small folios. He studied several Moorish-like tilings and investigated linked rings (seen in his last print, *Snakes*). He enumerated several tilings by congruent triangles while designing bank notes. He also recorded two theorems he evidently discovered but did not prove. One was about concurrent lines in a triangle, and the other about concurrent diagonals in a special tiling hexagon [53, pp. 82–93]. At my request, the first theorem was verified by A. Liu and M. Klamkin [35] and the second by J. F. Rigby [44].

An investigation in 1942 that was an amusement, shared with his children and grandchildren [18, pp. 9–11], [50], was combinatorial—to determine how many different patterns could be generated by following this algorithm:

> *Decorate a square with an asymmetric motif and use four copies of the decorated square (independently chosen from any of four rotated aspects) to fill out a 2×2 larger square, then translate the larger square in the direction of its edges to fill the plane.*

With a methodical search of the 4^4 possible 2×2 filled squares, eliminating obvious duplications, and by sketching examples with a simple motif for the rest, he ultimately found twenty-three distinct patterns. That is, no two of these twenty-three patterns were identical, allowing rotations. He also asked the combinatorial question in two special cases in which reflected aspects of the decorated square were also allowed. In these cases, the choices of the four copies of the decorated square were restricted as follows:

> Case (1)—two choices must be the same rotated aspect and independently, the other two choices the same reflected aspect. Case (2)—two choices must be different rotated aspects and independently, the other two choices different reflected aspects.

Escher's results were sketched in copybooks and later printed with inked carved wooden stamps using a motif that produced patterns resembling knitted or crocheted pieces. He also made a "ribbon" design, outlining crossing bands in a square, and carved four wooden stamps—one of the original design, one of its reflection, and two others that reversed the crossings in the first two stamps. He did not attempt to find the number of patterns produced by the 4^{16} possible 2×2 squares filled with aspects of these, but he did produce several patterns with them and colored them with a minimum number of colors so that continuous ribbon strands had the same color and no two bands of the same color ever crossed [53, pp. 44–52], [17, p. 41].

Escher's Interactions with Mathematicians

Until 1954 few mathematicians outside of Holland knew of Escher's work. That year the International Congress of Mathematicians (ICM) was held in Amsterdam, and N. G. de Bruijn arranged for an exhibit of Escher's prints, symmetry drawings, and carved balls at the Stedelijk Museum [11]. He wrote in the catalog, "Probably mathematicians will not only be interested in the geometrical motifs; the same playfulness which constantly appears in mathematics in general and which, to a great many mathematicians is the peculiar charm of their subject, will be a more important element" [2].

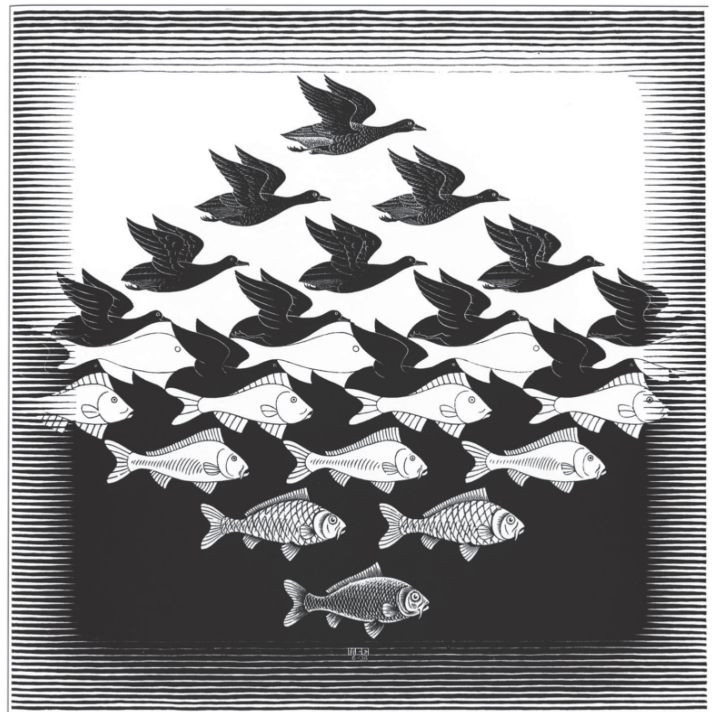

Figure 4. M. C. Escher's *Sky and Water I*, 1938. Woodcut, 435 cm x 439 cm.

When Roger Penrose visited the exhibit, he was amazed and intrigued. Escher's print *Relativity* especially caught his eye. It shows three prominent staircases in a triangular arrangement (and some smaller staircases), as seen from many different viewpoints, with several persons simultaneously climbing or descending them in an impossible manner, defying the law of gravity. Penrose was inspired to find a structure whose parts were individually consistent but, when joined, became "impossible". After returning to England he came up with the idea of the now-famous Penrose tribar in which three mutually perpendicular bars appear to join to form a triangle (Figure 5). Following that, his father devised an "endless staircase", another object that can be drawn on paper but is impossible to construct as it appears [41, pp. 149–50]. Penrose then closed the loop of discovery by sending the sketches of these impossible objects to Escher, who in turn used them in crafting the perpetual motion in his print *Waterfall* and the never-ending march of the monks in *Ascending and Descending*.

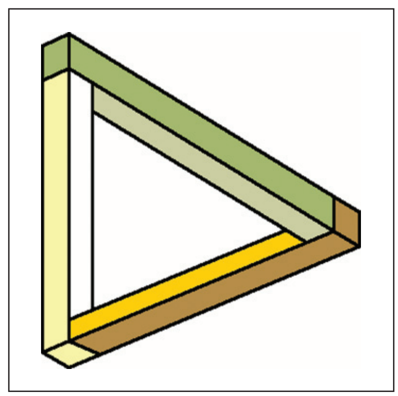

Figure 5. Penrose's tribar.

Penrose also visited Escher's home in 1962 and brought a gift of identical wooden puzzle pieces

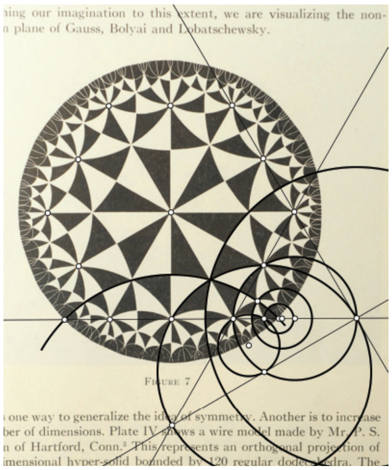

Figure 6. Coxeter's Figure 7, with Escher's markings (here computer-enhanced for visibility).

derived from a 60° rhombus. Escher soon sent Penrose the puzzle's solution, enclosing a sketch of the unique way in which the pieces fitted together. Here, congruent tiles were surrounded in two distinct ways. In 1971 Escher produced his only tiling with one tile that was not a regular division (today it would be called 2-isohedral). It was the last of his numbered symmetry drawings, with a little ghost that filled the plane according to the rules of Penrose's puzzle [41, pp. 144–45; 53, p. 229].

H. S. M. Coxeter also saw Escher's work for the first time during that ICM in 1954, and upon returning to Canada, he wrote Escher a letter to express his appreciation of the artist's work. Three years later, he wrote again to ask if he might use two of Escher's symmetry drawings to illustrate an article based on his presidential address to the Royal Society of Canada. The article discussed symmetry in the Euclidean plane and also in the Poincaré disk model of the hyperbolic plane and on a sphere surface [3]. Escher readily agreed, and when he later received a reprint of the article, he wrote to Coxeter, "some of the text-illustrations and especially figure 7, page 11, gave me quite a shock" [5, p. 19]. The figure's hyperbolic tiling, with triangular tiles diminishing in size and repeating (theoretically) infinitely within the confines of a circle, was exactly what Escher had been looking for in order to capture infinity in a finite space.

Escher worked over the figure with compass and straightedge and circled important points (Figure 6). From this, he managed to discern enough of the geometry to produce his print *Circle Limit I*. But he wanted to know more, and sent a large diagram to Coxeter showing what he had figured out, namely, the location of centers of six of the circles (Figure 7). In his letter, he politely asked Coxeter for "a simple explanation how to construct the [remaining] circles whose centres approach [the bounding circle] from the outside till they reach the limit." He also asked, "Are there other systems besides this one to reach a circle limit?" [5, p. 19], [54, p. 263]. Coxeter replied with a minimal answer to Escher's first request:

> The point that I have marked on your drawing (with a red o on the back of the page) lies on three of your circles with centres 1, 4, 5. These centres therefore lie on a straight line (which I have drawn faintly in red) and the fourth circle through the red point must have its centre on this same red line. [54, p. 264]

From this, Escher was supposed to construct the complete scheme. By contrast, Coxeter answered the second question at length, beginning, "Yes, infinitely many! This particular pattern is denoted by [4, 6]" and then explained for which p and q patterns $[p, q]$ exist, referring to the text

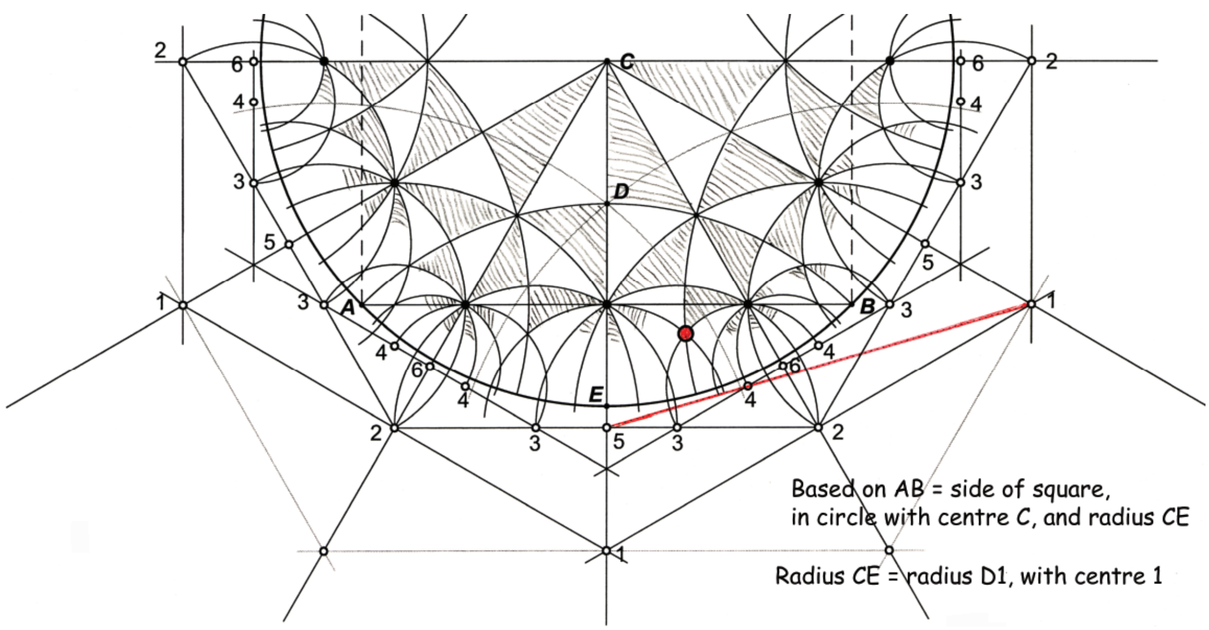

Figure 7. Escher's diagram sent to Coxeter, exhibiting what the artist had figured out. The original drawing is faint, drawn in pencil on tracing paper. This is a reconstruction by the author, and shows Coxeter's red markings.

Generators and Relations, and enclosing a "spare copy of $\left\{ {3 \atop 7} \right\}$" [54, p. 264].

Escher was disappointed with this reply, yet it only increased his determination to figure things out. He wrote to his son George:

> [Coxeter] encloses an example of using the values three and seven, of all things! However this odd seven is of no use to me at all; I long for two and four (or four and eight)… My great enthusiasm for this sort of picture and my tenacity in pursuing the study will perhaps lead to a satisfactory solution in the end. … it seems to be very difficult for Coxeter to write intelligibly to a layman. Finally, no matter how difficult it is, I feel all the more satisfaction from solving a problem like this in my own bumbling fashion. [1, p. 92], [54, pp. 264–5]

Escher did successfully carry out his "Coxetering", as he called his work with hyperbolic tilings, and in 1959–1960 he produced three other *Circle Limit* prints. Upon earlier receiving *Circle Limit I*, Coxeter had praised Escher for his understanding of the conformal pattern, and in 1960, when he received the complex *Circle Limit III*, Coxeter wrote Escher a three-page letter sprinkled with symbols explaining the print's mathematical content, with references to several technical texts, and the implications for coloring seen in the "compound {3, 8} [6{8, 8}] {8, 3} of six {8, 8}'s inscribed in a {3, 8}" [54, p. 265]. And Escher despaired to George, "Three pages of explanation of what I actually did… It is a pity that I understand nothing, absolutely nothing of it…" [1, pp. 100–01], [54, p. 265].

In 1960 Coxeter arranged for Escher to give two lectures at the University of Toronto about his work, and the Coxeters hosted the artist at their home. The Coxeter-Escher correspondence continued for several years, with two letters of note. In March 1964 Coxeter wrote "After looking again and again at your *Circle Limit III* on my study wall, I finally realized that my remark about its 'impossibility' was based on my own misunderstanding, as you will see in the enclosed," which was his review of Escher's book [20] for *Mathematical Reviews*. He added, "The more I look at your work, the more I admire it" [9]. That review [4] was the first time Coxeter revealed that the white arcs forming the backbones of fish in Escher's *Circle Limit III* were not, as he and others had assumed, badly rendered hyperbolic lines but rather were branches of equidistant curves. In 1979 and again in 1995 he published articles [5], [6] devoted to those white arcs, explaining, "they 'ought' to cut the circumference at the same angle, namely 80° (which they do, with remarkable accuracy). Thus Escher's work, based on his intuition, without any computation, is perfect…" [5, pp. 19–20].

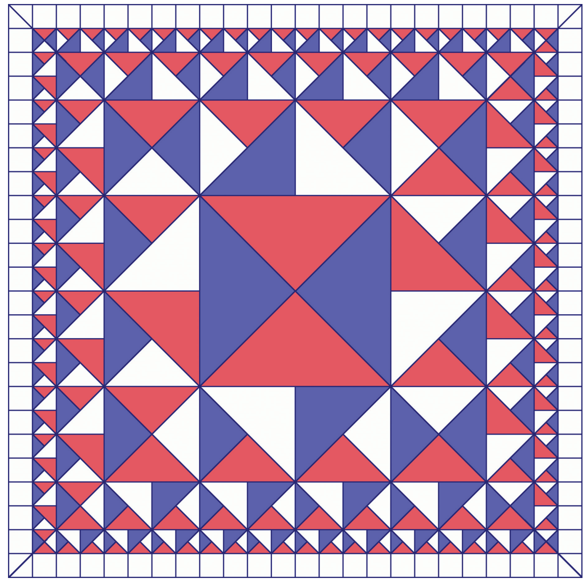

Figure 8. Escher's geometric grid for the print *Square Limit*.

In other articles Coxeter gave mathematical analyses of Escher's work and indicated that the artist had anticipated some of his own discoveries [54]. In May 1964 Escher sent Coxeter his print *Square Limit* and explained with a diagram its underlying geometric grid of self-similar triangles (see reconstruction in Figure 8). Escher's explanatory sketch was on graph paper, in red and blue colored pencil. It showed the first three rings surrounding the center square to indicate how the division process can continue forever. He had devised this fractal structure himself, and while a Euclidean construction with straight segments, it possessed the desired property of his *Circle Limits*—figures diminished as they approached the bounding square [17, pp. 104–05], [53, p. 315], [59, pp. 182–183]. A 90° rotation about the center of the diagram is a color symmetry, sending red tiles to blue, blue to red, and white to white. In Escher's print, the triangles are replaced by fish.

Escher had only brief interactions with other mathematicians; none would influence his work as did Pólya, Penrose, and Coxeter. Edith Müller, who had been A. Speiser's Ph.D. student, wrote to me that Escher had learned of her dissertation (a symmetry analysis of the Alhambra tilings) and visited her in 1948 in Zurich to discuss her (and his) work. She told him about how Speiser had learned to make lace in order to better understand symmetry.

Heinrich Heesch, another student of Speiser, carried out extensive research on tilings in the mid-1930s but did not publish until the 1960s. He, too, defined "regular" tilings as plane-fillings with congruent tiles in which every tile was surrounded in the same manner. Also, like Escher, he was interested in characterizing the conditions on edges of

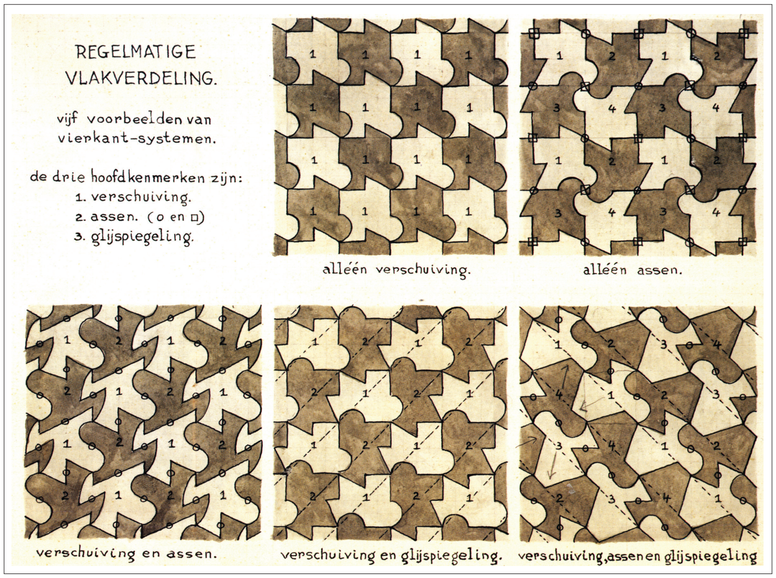

Figure 9. Escher's lecture slide about regular divisions of the plane, showing five "quadrilateral systems".

asymmetric tiles that could tile in this manner and for which the tiling had no reflection symmetries. He proved there were exactly twenty-eight types of these tiles and displayed a visual chart of them in his 1963 book with Otto Kienzle [29]. He assigned each edge of a tile a letter—*T*, *G*, or C_n—according to how it related to another edge by translation, glide-reflection, or $360°/n$ rotation. He sent the book to Escher, who at that time was very ill; for Escher, this information came more than twenty years after his own discoveries of all but one of those twenty-eight types [53, pp. 324–26].

In the last two years of Escher's life, mathematics teacher Hans de Rijk (a.k.a. Bruno Ernst) collaborated with Escher to write a book that would interpret the body of the artist's work, with special attention to the mathematical underpinnings of many prints. Every Sunday without fail they would spend time together as the manuscript took shape. This book [17] and a definitive catalog of Escher's graphic work [37] were both published in 1976, four years after the artist's death, and were the first to show many of Escher's painstaking preliminary drawings for his prints, some of them geometric marvels. A shorter version of de Rijk's analysis of Escher's work is in [1], pp. 135–54.

Escher's Work Used to Teach Mathematical Ideas

Escher enjoyed the role of teacher, giving lectures to diverse audiences—scientific gatherings, school students, museum audiences, even Rotary clubs. His lecture poster (Figure 9) shows in five different illustrative tilings how he explained the actions of translations (*verschuiving*), rotations (*assen*), and glide-reflections (*glijspiegeling*) that would carry a tile into an adjacent tile. Numbers identify the various aspects of a tile, circles and squares identify twofold and fourfold rotation centers, respectively, and adjacent dashed lines act as rails along which a tile glides and then reflects (in a line equidistant from the rails). Escher used large brightly-colored cardboard cutouts in the shapes of these tiles, mounted on straightened wire hangers, to demonstrate the motions of the isometries.

When Escher's book [20] was published in Holland in 1960, it included a short essay in the introduction by crystallographer P. Terpstra, to teach about symmetry and the seventeen plane symmetry groups. When the British translation was published, the essay appeared as a separate pamphlet; it never appeared with the American edition. Evidently the publisher, like Escher, thought it too technical.

Caroline MacGillavry, a crystallographer at the University of Amsterdam, was the first scientist to see the possibility of using Escher's art as a teaching tool in a text. When she first visited his studio in the late 1950s, she marveled: "The notebook in which he wrote his 'layman's theory' has been a revelation to me. It contains practically all the 2-, 3-, and 6-colour rotational two-dimensional groups, with and without glide-reflection symmetry" [39, p. *x*]. That visit gave birth to her idea of collaborating with Escher to use his symmetry drawings in a text for beginning geology students, to teach the classification of colored periodic tilings according to their symmetries. The International Union of Crystallography agreed to sponsor the publication. In the book's introduction, she notes,

> Escher's periodic drawings…make excellent material for teaching the principles of symmetry. These patterns are complicated enough to illustrate clearly the basic concepts of translation and other symmetry, which are so often obscured in the clumsy arrays of little circles, pretending to be atoms, drawn on blackboards by teachers of crystallography classes. On the other hand, most of the designs do not present too great difficulties for the beginner in the field. [39, p. *ix*]

In reviewing Escher's store of periodic drawings (by then, more than 100), she noted that one of the simplest symmetry groups, type *p2* with no color symmetries, was not represented. At her request, Escher produced a new symmetry drawing to fill the gap [39, plate 2], [53, p. 210]. He also produced another requested type [39, plate 34], [53, p. 211] and refreshed or redrew some others for the publication.

Coxeter may have been the first mathematician (outside of Holland) to use Escher's work to illustrate a mathematics text. His *Introduction to*

Geometry was unusual when it was published in 1961, with many nonstandard topics, including symmetry and planar tessellations, which he illustrated with Escher's symmetry drawings used earlier in [3]. Martin Gardner devoted a Mathematical Games column in *Scientific American* to a review of the book and republished the drawings, bringing Escher's symmetry work to the attention of the wider scientific world [21]. It was not long before scores of math texts (at all levels) and articles on teaching displayed Escher's periodic drawings and prints. While the elementary concepts of planar isometries, similarities, and symmetry are obvious ones for which Escher's symmetry drawings and prints provide wonderful illustrations, the drawings can also be used in teaching higher-level concepts of abstract algebra and group theory. In her article [57], Marjorie Senechal discusses how, by studying the color symmetry groups of Escher's periodic drawings, students can better understand the definition of a group, commutativity and noncommutativity, group action, orbits, generators, subgroups, cosets, conjugates, normal subgroups, stabilizers, permutations and permutation representations, and group extensions.

Teachers (and texts) of mathematics and science also use Escher's prints for artful depictions of mathematical objects (knots, Möbius bands, spirals, loxodromes, fractals, polyhedra, divisions of space) and to provide intriguing visual metaphors for abstract mathematical concepts (infinity, duality, reflection, relativity, self-reference, recursion, topological change) [49]. In his Pulitzer-Prize-winning book *Gödel, Escher, Bach: An Eternal Golden Braid*, Douglas Hofstadter uses Escher's work in essential ways to convey ideas of recursion and self-reference, and several authors have used Escher's prints to illustrate complex ideas of perception and illusion.

Often those who view art impose on it their reading of the artist's intention, and mathematicians' use of Escher's work to illustrate the idea of infinity and other mathematical concepts might be questioned. But it should be noted that Escher was intrigued by these concepts and set out to embody their essence in many of his prints. His fascination with infinity and how to capture it was a theme he returned to again and again. He spoke eloquently of this quest in his essay "Approaches to Infinity":

> Man is incapable of imagining that time could ever stop. For us, even if the earth should cease turning on its axis and revolving around the sun, even if there were no longer days and nights, summers and winters, time would continue to flow on eternally. …
>
> Anyone who plunges into infinity, in both time and space, further and further without stopping, needs fixed points, mileposts, for otherwise his movement is indistinguishable from standing still. There must be stars past which he shoots, beacons from which he can measure the distance he has traversed. He must divide his universe into distances of a given length, into compartments recurring in an endless sequence. Each time he passes a borderline between one compartment and the next, his clock ticks. …
> [37, pp. 37–40]

For Escher, mathematical concepts, especially infinity and duality, were a constant source of artistic inspiration.

Mathematical Research Related to or Inspired by Escher's Work

Several aspects of Escher's work anticipated by decades theoretical investigations by members of the scientific community. And some of his work has directly inspired mathematical investigations. We note here (necessarily briefly) many of these investigations.

Classification of "regular" tilings using edge relationships of tiles was Escher's method and also that of H. Heesch, but it was limited to asymmetric tiles and tilings with symmetry groups having no reflections. In the 1970s Branko Grünbaum and Geoffrey Shephard undertook a systematic classification of several kinds of tilings having transitivity properties with respect to the symmetry group of the tiling—isohedral (tile-transitive), isogonal (vertex-transitive), isotoxal (edge-transitive). Their method relied on using adjacency symbols and incidence symbols that recorded how (in the case of isohedral tilings) each tile was surrounded; the transitivity condition implied that every tile was surrounded in the same way. Their book [25] remains the fundamental reference on all aspects of tilings.

Two-color and 2-motif tilings were Escher's way of expressing duality. It is interesting to note that the first classification of two-color symmetry groups was carried out in 1936 (at almost the same time Escher was making his independent investigations) by H. J. Woods, who was interested in these black-white mosaics for textile designs [10], [62]. When a monohedral (one tile) tiling was colored in two colors, and a symmetry of the tiling interchanged the tiles and interchanged their colors, he called it "counterchange symmetry". (For example, in a checkerboard-colored tiling of the plane by squares, a reflection of the tiling in an edge of one column of squares would be a counterchange symmetry.) The scientific community and Escher were unaware of Woods's work. Later this kind of symmetry, so prevalent in Escher's work, was called "antisymmetry" by Russian crystallographers; that terminology is

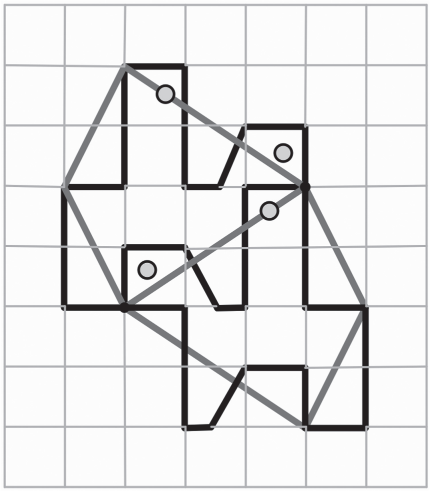

Figure 10. The beginning of *Horseman*.

not used today. Escher noted that some crystallographers had trouble accepting the idea of antisymmetry; he said that he couldn't work without it [1, p. 94].

Escher's method of splitting tiles to produce 2-motif tilings has been shown to be a powerful one. Today, the term 2-isohedral is used to describe tilings in which the symmetry group of the tiling produces two orbits of tiles—there are two distinct congruence classes of tiles with respect to the symmetry group of the tiling. It has been proved that every 2-isohedral tiling can be derived beginning with an isohedral tiling and applying the processes of splitting and gluing [13] and that this same process extends to produce *k*-isohedral tilings [32]. Andreas Dress [14] and I [55] have studied other aspects of these tilings.

Color symmetry was not a serious concern of crystallographers until the 1950s; even then, it was not easily embraced, and it took many years before color symmetry groups were studied systematically. When crystallographers and mathematicians did begin to investigate color symmetry groups, they (like Caroline MacGillavry) turned to Escher's work for illustrations and discoveries. Even today, there are competing notations for color symmetry groups [7], [25], [60], [61].

Metamorphosis, or *topological change,* was one of Escher's key devices in his prints. His interlocked creatures often began as parallelograms, squares, triangles, or hexagons, then seamlessly morphed into recognizable shapes, preserving an underlying lattice, as in his visual demonstration in Plate I in [19]. At other times the metamorphosis of creatures changed that lattice, as occurs in his *Metamorphosis III*. William Huff's design studio produced some intriguing examples of "parquet deformations" that preserve lattice structure [30], and, more recently, Craig Kaplan has investigated the varieties of deformation employed by Escher [34].

Covering surfaces with symmetric patterns was Escher's passion—the Euclidean plane, the hyperbolic plane, sphere surfaces, and cylinders—and always these coverings represented nontrivial symmetry groups of the patterned surface. Douglas Dunham has explored many families of Escher-like tilings of the hyperbolic plane and how to render them by computer [15], [52, pp. 286–296]. Others have studied how to cover different surfaces with periodic designs and sometimes asked, "What symmetry groups do these coverings represent?" See [7], [45], [56].

Escher's algorithm to produce patterns with decorated squares has inspired mathematicians and computer scientists to use combinatorial techniques (Burnside counting) and computer techniques to check his work and to answer more questions. Escher's results of twenty-three patterns for his simplest case and ten for his case (1) are exactly right. The correct answer for his case (2) is thirty-nine; for this case, Escher missed three patterns and counted one pattern twice [50]. Other questions have been asked and answered: How many patterns are there with two stamps (the original and its reflection) if Escher's restrictions on choice are removed [12]? How many with two stamps and translation only in one direction [42]? How many with the four "ribbon pattern" stamps and the additional action of under-over interchange added to the group of symmetries [22]? Can Escher's algorithm be computer-automated [38], [40]? Can allowable coloring of the ribbon patterns be automated [23]?

Creating tile shapes was almost an obsession with Escher. He would begin with a simple tile (often a polygon) that he knew would produce a regular division, then painstakingly coax the boundary into a recognizable shape. Who but Escher could conjure the polygon in Figure 10 into a helmeted horseman? [53, pp. 110–11] He explained,

> The border line between two adjacent shapes having a double function, the act of tracing such a line is a complicated business. On either side of it, simultaneously, a recognizability takes shape. But the human eye and mind cannot be busy with two things at the same moment and so there must be a quick and continual jumping from one side to the other. [39, p. *vii*]

Kevin Lee was the first to implement Escher's process with a computer program [36]. Craig Kaplan and David Salesin devised a computer program to address a complementary question—beginning with any shape, can it be gently deformed (still being recognizable) into a tile that will produce an isohedral tiling [33]?

Local vs. global definition of "regularity" was not Escher's concern; he followed the local rule that every tile be surrounded in the same way. But every one of Escher's "regular divisions" is an isohedral tiling; it satisfies the global regularity condition that the symmetry group is transitive on the tiles. An isohedral tiling necessarily has local regularity, but are the two definitions equivalent? In the Euclidean plane, yes, at least for asymmetric tiles and edge-to-edge tilings by polygons, but not so in the hyperbolic plane or in higher dimensions [51]. P. Engel also addresses this question in [16].

Symmetry of a tile inducing symmetry of its tiling was encountered and noted by Escher. When he used a tile with reflection symmetry (such as a dragonfly), it always induced reflections as symmetries of the tiling. He would note the tile was symmetric, and add an asterisk * to his classification symbol. But in a couple of instances, he created a tiling in which the tile was almost symmetric (and with slight modification can be made symmetric), yet the reflection line for the tile is not a reflection line for any of its tilings. In [24], Branko Grünbaum calls such tiles "hypersymmetric" and asks if they can be characterized. This is an open question.

Orderliness not induced by symmetry groups occurs at least twice in Escher's work: in his fractal construction of squares of diminishing size (Figure 8) and in his combinatorially perfect but not color-symmetry perfect coloring of one of his most complex designs with butterflies [1, p. 76]. Branko Grünbaum and others have asked for serious studies of other kinds of "orderliness" in tilings and patterns, not only that defined by symmetry groups [26], [27].

"*Completing*" *Escher's lithograph "Print Gallery"* recently posed a mathematical challenge to H. Lenstra and B. de Smit—how they came to understand the underlying geometric grid, "unroll" it, complete missing bits of the unrolled print, and roll it up again is described in [58].

In 1960 Escher wrote, "Although I am absolutely innocent of training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists" [20, Introduction]. Although he struggled with mathematics as a school student, when he became a graphic artist he was driven to pursue mathematical research, learn new geometric ideas, depict mathematical concepts, and pose mathematical questions. He could not have imagined the scope of influence his work would have for the scientific community.

Acknowledgments

The author thanks the M. C. Escher Company for permission to reproduce works by M. C. Escher. Bill Casselman took the photo of Coxeter's article used in Figure 6. Figures 3, 5, 6, 7, 8, and 10 were created by the author using *The Geometer's Sketchpad*.

References

[1] F. H. Bool, J. R. Kist, J. L. Locher, and F. Wierda, *M. C. Escher: His Life and Complete Graphic Work*, Harry N. Abrams, New York, l982; Abradale Press, 1992.

[2] N. G. de Bruijn, Preface, in *Catalog for the Exhibition M. C. Escher*, cat. 118, Stedelijk Museum, Amsterdam, 1954.

[3] H. S. M. Coxeter, Crystal symmetry and its generalizations, in A Symposium on Symmetry, *Trans. Royal Soc. Canada* **51**, ser. 3, sec. 3 (June 1957), 1–13.

[4] ______ , review of The Graphic Work of M. C. Escher, *Math. Rev.* MR0161210 (28:4418), 1964.

[5] ______ , The non-Euclidean symmetry of Escher's picture *Circle Limit III*, *Leonardo* **12** (l979), 19–25, 32.

[6] ______ , The trigonometry of Escher's woodcut *Circle Limit III*, in [52] pp. 297–305. Revision of *Math. Intelligencer* **18** no. 4 (1996), 42–46 and *HyperSpace* **6** no. 2 (1997), 53–57.

[7] ______ , Coloured symmetry, in [8], 15–33.

[8] H. S. M. Coxeter, M. Emmer, R. Penrose, and M. L. Teuber, eds. *M. C. Escher: Art and Science*, North-Holland, Amsterdam, 1986.

[9] H. S. M. Coxeter and M. C. Escher, Correspondence, M. C. Escher Archives, Haags Gemeentemuseum, The Hague, The Netherlands.

[10] D. W. Crowe, The mosaic patterns of H. J. Woods, *Comput. Math. Appl.* **12B** (1986), 407–411, and in *Symmetry: Unifying Human Understanding*, I. Hargittai, ed., Pergamon, New York, 407–411.

[11] J. Daems, Escher for the mathematician, Interview with N. G. de Bruijn and Hendrik Lenstra, *Nieuw Archief voor Wiskunde* **9**, no. 2 (2008), 134–137.

[12] D. Davis, On a tiling scheme from M. C. Escher, *Electron. J. Combin.* **4**, no. 2 (1997), #R23.

[13] O. Delgado, D. Huson, and E. Zamorzaeva, The classification of 2-isohedral tilings of the plane, *Geom. Dedicata* **42** (1992), 43–117.

[14] A. W. M. Dress, The 37 combinatorial types of regular "heaven and hell" patterns in the Euclidean plane, in [8], 35–46.

[15] D. J. Dunham, Creating repeating hyperbolic patterns—old and new, *Notices of the AMS* **50**, no. 4 (April 2003), 452–455.

[16] P. Engel, On monohedral space tilings, in [8], 47–51.

[17] B. Ernst (J. A. F. de Rijk), *The Magic Mirror of M. C. Escher*, Random House, New York, l976; Taschen America, 1995.

[18] G. Escher, M. C. Escher at work, in [8], 1–11.

[19] M. C. Escher, The regular division of the plane, in [1], pp. 155–173 and in [31], 90–127.

[20] ______ , *Grafiek en Tekeningen M. C. Escher*, J. J. Tijl, Zwolle, 1960. *The Graphic Work of M. C. Escher*, Duell, Sloan and Pearce, New York, 1961; Meredith, 1967; Hawthorne, 1971; Wings Books, 1996.

[21] M. Gardner, Concerning the diversions in a new book on geometry, *Sci. Amer.* **204** (1961) 164-175. In *New Mathematical Diversions*, MAA, 2005, pp. 196-209.

[22] E. Gethner, D. Schattschneider, S. Passiouras, and J. Joseph Fowler, Combinatorial enumeration of 2×2 ribbon patterns, *European J. Combin.* **28** (2007), 1276–1311.

[23] E. Gethner, Computational aspects of Escher tilings, Ph.D. dissertation, University of British Columbia, 2002.

[24] B. Grünbaum, Mathematical challenges in Escher's geometry, in [8], 53–67.

[25] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1987.

[26] B. Grünbaum, Levels of orderliness: Global and local symmetry, in *Symmetry 2000*, I. Hargittai and T. C. Laurent, eds., Portland Press, London, 2002, 51–61.

[27] ______ , Periodic ornamentation of the fabric plane: Lessons from Peruvian fabrics, in *Symmetry Comes of Age: The Role of Pattern in Culture*, D. K. Washburn and D. W. Crowe, eds., U. of Washington Press, Seattle, 2004, 18–64.

[28] F. HAAG, Die regelmässigen Planteilungen und Punktsysteme, *Z. Krist.* **58** (1923), 478–488.

[29] H. HEESCH and O. KIENZLE, *Flächenschluss. System der Formen lückenlos aneinanderschliessender Flachteile*, Springer, Berlin, 1963.

[30] D. HOFSTADTER, Parquet deformations: Patterns of tiles that shift gradually in one dimension, *Scientific American* (July 1983): 14–20. Also in *Metamagical Themas: Questing for the Essence of Mind and Pattern*, Basic Books, New York, 1985, 191–212.

[31] W. J. VAN HOORN and F. WIERDA, eds., *Escher on Escher: Exploring the Infinite*, Harry N. Abrams, New York, 1989.

[32] D. H. HUSON, The generation and classification of *k*-isohedral tilings of the Euclidean plane, the sphere, and the hyperbolic plane, *Geom. Dedicata* **47** (1993), 269–296.

[33] C. S. KAPLAN and D. H. SALESIN, Escherization, in *Proc. 27th Inter. Conf. Computer Graphics and Interactive Techniques (SIGGRAPH)*, ACM Press/Addison Wesley, New York, 2000, 499–510.

[34] C. S. KAPLAN, Metamorphosis in Escher's art, *Bridges Leeuwarden Conf. Proc. 2008*, Tarquin, 39–46.

[35] M. S. KLAMKIN and A. LIU, Simultaneous generalizations of the theorems of Ceva and Menelaus, *Math. Mag.* **65** (1992), 48–52.

[36] K. D. LEE, Adapting Escher's rules for 'regular division of the plane' to create *TesselMania!*®, in [52], 393–407.

[37] M. C. ESCHER, Approaches to infinity, in *The World of M. C. Escher*, J. L. Locher, ed., Harry N. Abrams, New York, 1972.

[38] R. MABRY, S. WAGON, and D. SCHATTSCHNEIDER, Automating Escher's combinatorial patterns, *Mathematica in Ed. and Res.* **5**, no. 4 (1996–97), 38–52.

[39] C. H. MACGILLAVRY, *Symmetry Aspects of M. C. Escher's Periodic Drawings*, Oosthoek, Utrecht, 1965. Reprinted as *Fantasy and Symmetry*, Harry N. Abrams, New York, 1976.

[40] S. PASSIOURAS, *Escher Tiles*, `http://www.eschertiles.com/`

[41] R. PENROSE, Escher and the visual representation of mathematical ideas, in [8], 143–157.

[42] T. PISANSKI, B. SERVATIUS, and D. SCHATTSCHNEIDER, Applying Burnside's lemma to a one-dimensional Escher problem, *Math. Mag.* **79**, no. 3 (2006), 167–180.

[43] G. PÓLYA, Über die Analogie der Kristallsymmetrie in der Ebene, *Z. Krist.* **60** (1924), 278–282.

[44] J. F. RIGBY, Napoleon, Escher, and tessellations, *Math. Mag.* **64** (1991), 242–246.

[45] D. SCHATTSCHNEIDER and W. WALKER, *M. C. Escher Kaleidocycles*, Ballantine, New York, 1977; Pomegranate Communications, Petaluma, 1987; Taschen, Berlin, 1987.

[46] D. SCHATTSCHNEIDER, M. C. Escher's classification system for his colored periodic drawings, in [8], pp. 82–96, 391–392.

[47] ______ , The Pólya-Escher connection, *Math. Mag.* **60** (1987), 292–298.

[48] ______ , Escher: A mathematician in spite of himself, *Structural Topology* **15** (1988) 9–22. Reprinted in *The Lighter Side of Mathematics*, MAA, 1994, 91–100.

[49] ______ , Escher's metaphors: The prints and drawings of M. C. Escher give expression to abstract concepts of mathematics and science, *Sci. Amer.* **271**, no. 5 (1994), 66– 71.

[50] ______ , Escher's combinatorial patterns, *Electron. J. Combin.* **4**, no. 2 (1997), #R17.

[51] D. SCHATTSCHNEIDER and N. DOLBILIN, One corona is enough for the Euclidean plane, in *Quasicrystals and Discrete Geometry*, J. Patera, ed., Fields Inst. Monographs, v. 10, AMS, 1998, 207–246.

[52] D. SCHATTSCHNEIDER and M. EMMER, eds., *M. C. Escher's Legacy: A Centennial Celebration,* Springer-Verlag, 2003.

[53] D. SCHATTSCHNEIDER, *M. C. Escher: Visions of Symmetry*, W. H. Freeman, 1990, new edition Harry N. Abrams, 2004.

[54] ______ , Coxeter and the artists: Two-way inspiration, in *The Coxeter Legacy: Reflections and Projections*, C. Davis and E. W. Ellers, eds., Fields Inst. Comm., ser. no. 46, AMS, 2006, 255–280.

[55] ______ , Lessons in duality and symmetry from M. C. Escher, in *Bridges Leeuwarden Conf. Proc. 2008*, Tarquin, 2008, 1–8.

[56] M. SENECHAL, Escher designs on surfaces, in [8], 97–110.

[57] ______ , The algebraic Escher, *Struct. Topology* **15** (l988), 31–42.

[58] B. DE SMIT and H. W. LENSTRA JR., The mathematical structure of Escher's *"Print Gallery"*, *Notices of the AMS* **50**, no. 4 (2003), 446–451. Also `http://escherdroste.math.leidenuniv.nl`.

[59] E. THÉ (design), *The Magic of M. C. Escher*, Harry N. Abrams, New York, 2000.

[60] D. K. WASHBURN and D. W. CROWE, *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, U. of Washington Press, Seattle, 1988.

[61] T. W. WIETING, *The Mathematical Theory of Chromatic Plane Ornaments*, Marcel Dekker, New York, 1982.

[62] H. J. WOODS, Counterchange symmetry in plane patterns, *J. Textile Inst.* **27** (1936), T305–T320.